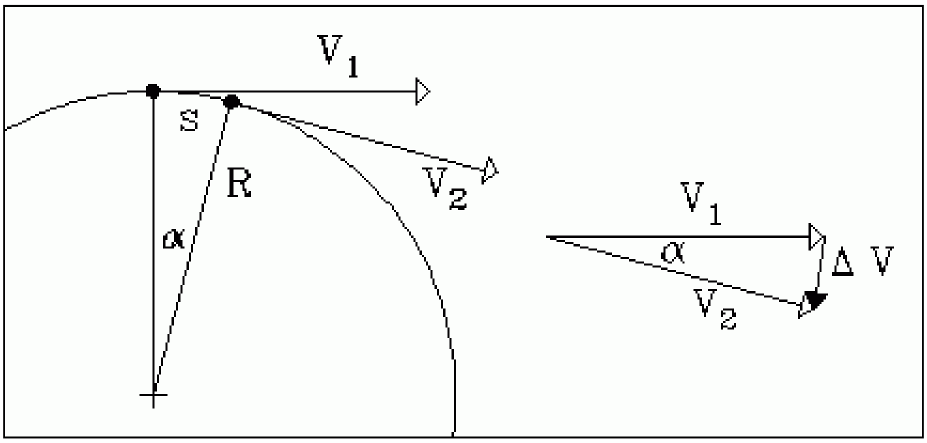
Centripetal force, weight, stress and the Earth's equatorial bulge.

by Donald E. Simanek

> ***Disclaimer:*** *This document, like many others on this website, attempts to get across a complex bit of physics in language as simple, conceptual and non-mathematical as possible. This is, of course, the hard way to do it. If we were to use full-blown vector calculus the arguments could be a lot more compact, and far more precise. The reader is strongly urged to consult such treatments in textbooks.*

Centripetal Force.



The acceleration of a body is the change in its velocity divided by the time duraton of that change, $\boldsymbol{a} = \Delta V/\Delta t$. Since velocity is a vector, any change in its size **or direction**, or both, requires acceleration. A body moving in a curved path is accelerating, even if its speed is constant in size. Therefore, from Newton's law, we know that the net force on it is non-zero.

The diagram shows a body moving with constant speed, V on a curved path of radius R. Two positions are shown, during which the body has moved a distance S along an arc. At the beginning of the time interval the body's velocity is $\boldsymbol{V_1}$. At the end of the interval it is $\boldsymbol{V_2}$. We give them distinguishing subscripts because they have different directions, even though they have the same size. During that time the body moves through angle α. At the right we show a vector diagram of the relation between these velocities and their vector difference, $\Delta \boldsymbol{V}$.

Now consider the limiting case as the time interval gets very small, approaching zero. The angle approaches zero also. The diagrams have two similar, very skinny triangles. We can write:

$\Delta S / R = \Delta V / V$

So: $V \Delta S = R \Delta V$

and: $V (\Delta S / \Delta t) = R (\Delta V / \Delta t)$.

But $V = \Delta S / \Delta t$, and $a = \Delta V / \Delta t$, so we can write:

$V^2 = R a$, which becomes $a = V^2/R$.

This is the **size** of the centripetal acceleration vector. The direction of the centripetal acceleration vector is inward toward the center of the arc at any instant, and is therefore perpendicular to the velocity vector at that instant, which is always tangent to the curve.

We can associate this acceleration with the inward (radial) component of whatever net force happens to be acting on the body (of mass m). We call that component the **centripetal force**, with size $F_{centripetal} = m a = mV^2/R$. It, too, is a vector directed radially inward.

Centripetal force is not some new kind of force, but just a convenient name for the radial component of the sum of all of the real forces acting on the body.

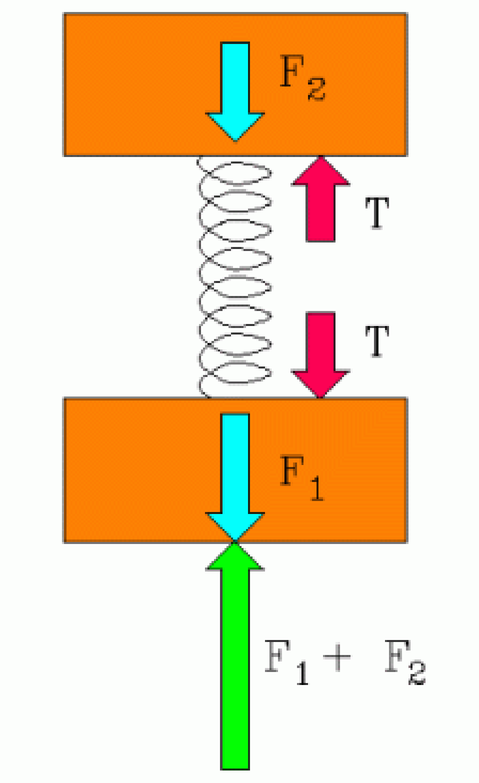
A fuller treatment would show that this definition of centripetal force is useful for any kind of motion along a curve of any sort. It is not restricted to circles. Nor is it restricted to constant speed along the path. This works because any physical path is such that a small enough portion of it approximates a circlular arc very well. In calculus, we take the limit as the arc becomes zero length, and speak of the relation between the instantaneous radius of the arc at that point on the curve, the instantaneous velocity at that point and the instantaneous acceleration there. The relation still turns out to be $a = V^2/R$. And if the net force has a component tangent to the arc, that causes an increase or decrease in the body's speed along that arc.

For a calculus treatment of this and many other physics topics, consult Jess H. Brewer's excellent physics tutorial, The Skeptic's Guide to Physics. See the celestical mechanics chapters.

Stress.

Elastic bodies can deform under applied forces. The bonding forces which hold molecules in an equilibrium position in solid and liquid bodies act something like springs. If a spring is stretched by pulling molecules farther apart, these bonding forces increase in size. They also increase in size when the molecules are pushed closer together. We can model this behavior as shown in the figure, with a spring representing such forces acting between chunks of matter. For our limited purposes, we may assume that the spring is approximately Hookian in behavior, that is, it obeys Hooke's law. The law says that the spring tension T changes by amount $\Delta T = - K\Delta L$ where k is the spring's elastic constant and ΔL is the change in its length, measured from its initial position. If the tension increases, the spring's length decreases. If the tension decreases, the spring's length increases.

We will assume the mass of the spring itself is negligible compared to the mass of the two blocks of matter. Therefore the spring tension exerts forces of equal size on each of them, as shown.

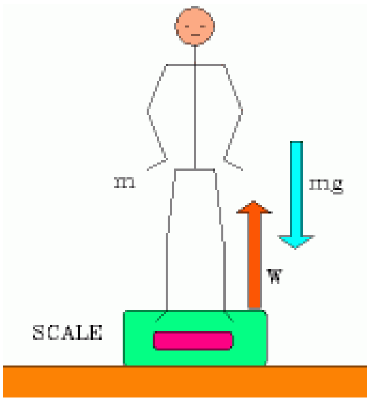


F_2 and F_1 represent the gravitational force on upper and lower blocks. In equilibrium the lower block must experience an upward force of $F_1 + F_2$ (from whatever supports it from below). If these were equal in size, then for equilibrium, the tension T would also be that same size.

Now suppose that the upper block were made smaller in mass. The tension in the spring must get proportionally smaller to achieve static equilibrium. Therefore the spring must increase in length, separating the two blocks a bit.

The same thing would happen if some additional non-uniform gravitational field were applied to the system, such that it exerted greater downward force on the lower block than it did on the upper one. The tension on the spring would decrease and its length would increase. But what if such an external field caused a stronger upward force on the upper block than on the lower one? Again, the tension on the spring would decrease and its length would increase. This becomes important when one discusses Earth tides due to non-uniform gravitational fields from the Moon.

Weight.



Weight may be defined as the force required to keep an object at rest relative to its surroundings. This definition is consistent with most colloqual interpretations of the word (surprise!).

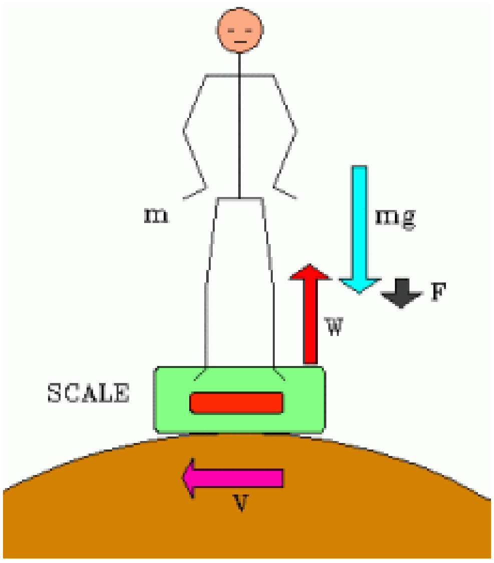
The figure shows a person of mass m standing on a bathroom scale on the surface of the Earth. The scale exerts a force of size W upward on his feet. We call this scale reading the "weight" of the man. If the Earth were stationary, the man would be in equilibrium, and we would have $W = mg$, where mg is the gravitational force on the man.

Think of the scale as like the spring between masses. It responds to the stress between feet and floor.

Effect of Earth rotation on weight.

But if the Earth is rotating, the man, scale and ground under it are moving with speed V in a circular path of radius R, where R is the radius of the Earth. These objects are no longer in static equilibrium. They all have an acceleration $a = v^2/R$ toward the center of the Earth. This centripetal acceleration changes the direction of velocity, but doesn't change its size, because the acceleration vector is always directed toward the center of the circle, and is therefore perpendicular to the body's velocity at any time.

The size of the man relative to the Earth's radius is greatly exaggerated in this diagram. The relative sizes of the forces are also.



Newton's law tells us that when a body of mass m has an acceleration ***a***, the net force on the body must be ***F*** = ***ma***. So the size of the net force on this man must be $F = ma = mv^2/R$.

The figure shows the force vectors ***W*** and ***mg***, which are the only forces acting on the man. The vector ***F*** is their sum. ***W*** is directed along the radius of the Earth. Being the radial component of the net force (it **is** the net force in this case), its size is $a = v^2/R$ (the centripetal force). Now compare these two cases. On the non-rotating Earth the man's weight was of size ***mg***. Remember, the weight of an object is the force required to support it, i.e., the force exerted upward by the weighing scale. With the Earth rotating, that force is **smaller** than before. The contact force between the man's feet and the scale is reduced. But all other such stress forces are reduced as well, within the man, within the scale's springs, within the body of the Earth itself. This causes a slight decompression of these materials, a relaxation of the spring in the scales. In fact, the entire body of the earth expands slightly and the man and scale move outward from the axis of rotation slightly, until forces come into balance with the requirements of rotational stability at the new radius. This is the reason for the equatorial bulge of the Earth due to its own axial rotation.

As we said, the diagrams are exaggerated. Simple calculation shows that the centripetal acceleration is only 0.3% of *g*. So the net force is only 0.3% of *W*. The man's weight (registered on the scale) is 0.3% smaller than it would be at one of the Earth's poles. The resulting relaxation of stress in materials is the reason for the equatorial bulge of the Earth, making the equatorial radius 43 kilometer greater than the polar radius.

The "cause and effect" dichotomy.

Here we can get into a tricky dilemma of interpretation. Is the weight reduction at the equator simply due to the smaller gravitational force at the greater radius? Or is it simply due to the reduction in stress between his feet and the scale? This is another unfortunate consequence of using "cause and effect" language in something that is too complex for such a simplistic description. It is also a trap for teachers who would pose multiple choice questions on exams.

Suppose someone says that the reduction in weight is solely due to the increased radius of the equator and the slightly smaller gravitational field at the larger radius. Then if the earth were prefectly rigid and there were no equatorial bulge, you would logically conclude from this that since the radius doesn't change, then the weight (registed on the scale) doesn't change. But this happens to be false. The weight would decrease, as it must, from the fact of the radial acceleration due to the circular motion and Newton's law $\boldsymbol{F} = m\boldsymbol{a}$.

If someone says that the reduction in weight is due to the decreased stress caused by the stretching of the materials of the Earth, scale and man, we must agree, for that's what the scale mechanism measures, and that's our definition of weight. But this stress reduction is also the reason for the stretch of the earth, and for the increased radius at the equator. And this in turn does decrease the gravitational force due to the Earth at that larger distance. The bottom line is that the gravitational force decreases and the stress decreases, **but not in proportion**. To say just one of these things is the "cause" of any one of the others is just too simplistic to be useful. And to try to "get by" explaining the equatorial bulge without mentioning stress reduction in materials is somewhat of a cheat.

Is "centrifugal force" necessary here?

Notice that in this discussion we never had to introduce the term "centrifugal force". That term is only useful when one chooses to analyze this problem in a rotating non-inertial coordinate system.

Non-inertial coordinate systems

An "inertial" coordinate system is one that isn't accelerating. How can we be sure a coordinate system isn't accelerating, when everything in the universe seems to be moving, and accelerating under gravitatonal forces from other bodies? One way is to identify all real forces acting on the body. If their vector sum on the body is zero, then that body isn't accelerating. Another definition of "inertial frame" is a frame of reference such that a body fixed in that frame strictly obeys Newton's second law, $\boldsymbol{F} = m\boldsymbol{a}$.

But inertial frames are very rare. We do encounter many situations where "for all practical purposes" a system seems to be an inertial frame. On the surface of the earth, a laboratory can often be considered an inertial frame, for the net acceleration of the room as it is carried around by the earth's axial rotation and its revolution around the sun, and by the solar systems motion in the galaxy, etc., is very small compared to the larger accelerations we are studying. We do include the effect of axial rotation by "correcting" the gravitational force to include the centripetal acceleration. And if we are dealing with very large scale motions of air and water, we must include also the Coriolis effects due to the acceleration of our reference frame.

What this boils down to is that if the net force on a body in our reference frame $\boldsymbol{F_{net, real}} = \boldsymbol{ma} + \boldsymbol{F_{extra}}$, we try to identify that extra force term, and then subtract it from the "real" forces, and go ahead and use this "corrected" form of Newton's laws. This is equivalent to defining $\boldsymbol{-F_{extra}}$ as "fictitious" force, adding that to the real force term, and thereby preserving Newton's law, using it "as if" we were doing the problem in an inertial frame.

In a frame of reference centered at the earth's center and rotating with the earth, one of these fictitious forces on a body at the earth's surface is called the "centrifugal" force. It is just the negative of the centripetal force on that body. It is directed outward from the earth's center. Other important forces at the earth's surface are the coriolis forces, significant when dealing with huge masses of air or water in meteorology, oceanography, and long range ballistics.

This lengthy preamble is leading up to an often-asked question. "When we use the term centripetal force, does that mean we are doing the problem in a non-inertial frame?" No, it doesn't. Quite the contrary, for centripetal force on an object is just the radial component of the real force on it. "Centripetal force" is used primarily when we are doing a problem in which something is moving in an orbit around a fixed point, and we are using a fixed (inertial) coordinate frame centered on that point. We can know that if the the vector sum of all forces acting on a body fixed in that reference frame do add to zero. Planetary orbits are a good example. Generally in these cases, we are using an inertial polar coordinate frame of reference.

But when we choose to fix our coordinate system on a body we know is accelerating, so that the coordinates are also accelerating (perhaps undergoing both rotation and linear acceleration), then we find that the real forces we measure acting on a body fixed in that (moving) frame do **not** add to zero.

Much of the confusion about tides stems from failure to specify whether we assume an inertial frame of reference, or an accelerting frame of reference. It can also arise from forgetting that inertial frames can be either cartesian or polar. Polar coordinates are not limited to rotating frames.

Appendix and Summary

Questions about centripetal and centrifugal force are sure to provoke controversy amongst physics teachers, and I have strong feelings on it myself.

The Wikepedia has two pages worth looking at: Centrifugal force. Be sure to take the link to "Centrifugal force (rotating reference frame)".

Centripetal force is defined to be the radial component of the net force on a body when the body's position is represented in a polar coordinate system (coordinates being radius from a fixed center and angle from a fixed reference angle). It is just a label to distinguish the radial component from the tangential component of the net force. But it is a useful name, for the radial component figures into a very useful equation: $F = mv^2/r$.

Centripetal force, being a component of the net force on a body, is a "real" force. Real forces are those forces acting on a body due to ***other*** material objects: gravitation, electric attraction and repulsion, magnetic forces, contact forces (deformation from contact, and also forces due to friction and rolling resistance).

In high school textbooks one sometimes sees centrifugal force "defined" as the reaction force to the centripetal force. I deplore this misleading idea with a passion, for it is not necessary and not useful for anything. Furthermore it causes anguish and confusion when students go to college and learn the definition of centrifugal force used when dealing with rotating frames of reference.

Rotating reference frames are useful, employed by mechanical engineers and astronomers or anyone who must do actual calculations in rotating reference frames. Centrifugal force is a fictitious force used to simplify such calculations. But this approach is seldom used in introductory courses, so it would be best not to mention it there. Yet it is appealing to naive students (and to some textbook authors) because it seems to correspond to the feeling of being "thrown outward" when one is on a rotating platform, as on a carousel, or in an automobile going around a tight curve. It's a "feel-good" explanation for students who will not have to actually do any calculations using the concept. Actually, what a person feels in a situation such as this is the result of the car exerting a force perpendicular to your motion, that causes your own motion to depart from straight line motion into a curved path. It is a physiological feeling. But it does, of course, arise from real forces of pressure in your body, stimulating nerves. But these indicate only the size of forces, not their direction.

Centrifugal force is not a real force on a body due to any other physical object. That's why we call it a "fictitious force".

Any problem done in a rotating coordinate system can be done in an fixed coordinate system without using the centrifugal force concept. The results, by the two methos, must, of course, be identical.

A polar coordinate system can be a fixed, inertial coordinate system.

A rotating coordinate system can be either polar or cartesian. "polar" and "rotating" are not synonyms.

If your chosen coordinates are not rotating, the term "centrifugal force" is not appropriate. If you don't know whether your coordinates are rotating, then you need to study the basics to find out, for if you proceed without knowing that fact you have a 50-50 chance of messing up the problem, and any attempt to discuss it will result in confusion for everyone.

Newton's $\boldsymbol{F} = m\boldsymbol{a}$ applies only in inertial (fixed, non-accelerating) coordinate systems.

Put another way, the word "inertial" applies to any coordinate system in which Newton's law is correct. It also means "any coordinate system that isn't accelerating". A coordinate system moving with acceleration is a non-inertial system.

When you do a problem in a rotating non-inertial system, you must modify Newton's law to read $\boldsymbol{F}_{real} + \boldsymbol{F}_{centrifugal} = m\boldsymbol{a}$, where $F_{\text{centrifugal}}$ is a fictitious "correction" force to compensate for using a non-inertial coordinate system, and is **not** a force due to the physical influence of other real objects.

Many discussions I see of this on the internet are often ill-considered opinions that people offer without having looked at the broader picture. Indeed, many teachers have not had sufficient exposure to this in college and university. It generally occupies a chapter in a university classical mechanics course, titled "Non-inertial systems". A good treatment can be found in the excellent (and classic) book "Classical Dynamics of Particles and Systems" by Jerry marion (Academic Press, 1965, and later editions revised by Davidson). See chapter 12, "Motion in a Non-Inertial Reference Frame". Any good university library should have this. In the United States (I hate to admit) most high school teachers have never taken such a course, so are entirely innocent of the standard methods for doing problems in the non-inertial frame of the earth (necessary in ocean hydrodynamics and meteorology as well as in long range ballistics of rockets missiles and spacecraft) and in astrophysics. Engineers need these methods when dealing with gyroscopic effects.

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