A

MANUAL

OF

SPHERICAL AND PRACTICAL

ASTRONOMY:

EMBRACING

THE GENERAL PROBLEMS OF SPHERICAL ASTRONOMY, THE SPECIAL
APPLICATIONS TO NAUTICAL ASTRONOMY, AND THE THEORY
AND USE OF FIXED AND PORTABLE ASTRO-
NOMICAL INSTRUMENTS.

WITH AN APPENDIX ON THE METHOD OF LEAST SQUARES.

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VOL. I.

SPHERICAL ASTRONOMY.

PREFACE.

THE methods of investigation adopted in this work are in accordance with what may be called the modern school of practical astronomy, or more distinctively the German school, at the head of which stands the unrivalled BESSEL. In this school, the investigations both of the general problems of Spherical Astronomy and of the Theory of Astronomical Instruments are distinguished by the generality of their form and their mathematical rigor. When approximative methods are employed for convenience in practice, their degree of accuracy is carefully determined by means of exact formulæ previously investigated; the latter being developed in converging series, and only such terms of these series being neglected as can be shown to be insensible in the cases to which the formulæ are to be applied. And it is an essential condition of all the methods of computation from data furnished by observation, that the errors of the computation shall always be practically insensible in relation to the errors of observation : so that our results shall be purely the legitimate deductions from the observations, and free from all avoidable error.

It is another characteristic feature of modern spherical astronomy, that the final formulæ furnished to the practical computer are so presented as seldom to require accompanying verbal precepts to distinguish the species of the unknown angles and arcs; and this results, in a great measure, from the consideration of *the general spherical triangle*, or that in which the six parts of the triangle are not subjected to the

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condition that they shall each be less than 180°, but may have any values less than 360°, all ambiguity as to their species being removed by determining them, when necessary, by two of their trigonometric functions, usually the sine and the cosine. This feature is mainly due to GAUSS, and was prominently exhibited in his *Theoria Motus Corporum Cœlestium*, published in 1809. The English and American astronomers have been slow to adopt this manifest improvement; in evidence of which I may remark that the general spherical triangle was not treated of in any work in the English language, so far as I know, prior to the publication of my *Treatise on Plane and Spherical Trigonometry*, in the year 1850. In the present work, I assume the reader to be acquainted with this form of spherical trigonometry, and to accept its fundamental equations in their utmost generality.

A third and eminently characteristic feature of modern astronomy, is the use which it makes, in all its departments, of the method of least squares, namely, that method of combining observations which shall give the *most probable* results, or which shall be exposed to the least probable errors. This method is also due to GAUSS, who (though anticipated in the publication of one of its practical rules by LEGENDRE) was the first to give a philosophical exposition of its principles. The direct effect of this improvement is not only that the most probable result in each case is obtained, but also that the relative degree of accuracy of that result is determined, and thus the degree of confidence with which it may be received and the weight which it may be allowed to have in subsequent discussions. Judiciously employed, it serves to indicate when a particular process has reached the limit of accuracy which it can afford, thereby saves fruitless labor, directs inquiry into new channels, and contributes greatly to accelerate the progress of the science.

Whilst the science has been rapidly advancing in Europe, we have in this country not been idle. Two of the most important improvements in practical astronomy have had

than usual completeness. The fundamental formulæ adopted are those of BESSEL'S theory, but the solutions of the various problems relating to the prediction of solar eclipses for the earth generally are mostly new. The rigorous solutions of these problems given by BESSEL in his *Analyse der Finsternesse* are not required for the usual purposes of prediction, however interesting they may be as specimens of refined and elegant analysis. On the other hand, the approximate solutions commonly given appear to be unnecessarily rude. Those that I have substituted will be found to be very little if at all more laborious than the latter, while they are almost as precise as the former, and by a very little additional labor (that is, by repeating only some parts of the computation for a second or third approximation) may be rendered quite exact.

So far as I can find, no one has heretofore treated distinctively of the occultations of *planets* by the moon, and these phenomena have been dismissed as simple cases of the general theory of eclipses, in which both the occulting and the occulted body are spherical. But in almost every occultation of one of the principal planets, the planet will be either a spheroidal body fully or partially illuminated by the sun, or a spherical body partially illuminated: so that, in the general case, we have to consider the disc of the occulted body as bounded by an ellipse or by two different semi-ellipses. I have discussed this general case at length, and have adapted the theory to each planet specially. The additional computations required to take into account the true figure of the planet's disc are sufficiently brief and simple. The case of the occultation of a cusp of Venus or Mercury is included in the discussion, and also the occultation of Saturn's rings.

The well known formula for predicting the transits of the inferior planets over the sun's disc, first given by LAGRANGE, is here rendered more accurate by introducing a considera-

to the rejection of nearly the same observations as that of PEIRCE.

The plates at the end of the work exhibit in minute detail the instruments now chiefly employed by astronomers. To have given more, with the necessary explanations, would have led me too far into the mere history of the subject, and would have occupied space which I thought it preferable to fill with discussions relating to the leading instruments now in use. The scale of these plates is purposely made quite small; but the great precision with which they are executed will enable the reader to measure from them the dimensions of all the important parts of each of the principal instruments. I am greatly indebted for the perfection of these drawings to the engravers, the Messrs. ILLMAN BROTHERS, of Philadelphia.

Such auxiliary tables as seemed to be necessary to the reader in using these volumes have been given at the end of Vol. II. Some of these are new. Most of those which have been derived from other sources have been either re-computed or tested by differences and corrected. To insure their accuracy, they have also been tested by differences after being in type.

For the very complete index to the whole work, I am indebted to my friend, Prof. J. D. CREHORE, of Washington University.

In conclusion, I desire to express my obligations to those citizens of Saint Louis who, without solicitation, have generously assumed a share of the risk of publication. Their liberal spirit has been met by a corresponding liberality on the part of my publishers, who have spared no expense in the typographical execution. I shall be content if their expectations are not wholly disappointed, and the work contributes in any degree to the advancement of the noblest of the physical sciences.

WASHINGTON UNIVERSITY,
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SPHERICAL ASTRONOMY.

CHAPTER I.

THE CELESTIAL SPHERE—SPHERICAL AND RECTANGULAR CO-ORDINATES.

1. FROM whatever point of space an observer be supposed to view the heavenly bodies, they will appear to him as if situated upon the surface of a sphere of which his eye is the centre. If, without changing his position, he directs his eye successively to the several bodies, he may learn their relative directions, but cannot determine either their distances from himself or from each other.

The position of an observer on the surface of the earth is, however, constantly changing, in consequence, 1st, of the diurnal motion, or the rotation of the earth on its axis; 2d, of the annual motion, or the motion of the earth in its orbit around the sun.

The changes produced by the diurnal motion, in the apparent relative positions or directions of the heavenly bodies, are different for observers on different parts of the earth's surface, and can be subjected to computation only by introducing the elements of the observer's position, such as his latitude and longitude.

But the changes resulting from the annual motion of the earth, as well as from the proper motions of the celestial bodies themselves, may be separately considered, and the directions of all the known celestial bodies, as they would be seen from the centre of the earth at any given time, may be computed

according to the laws which have been found to govern the
motions of these bodies, from data furnished by long series of
observations.   The complete investigation of these changes and
their laws belongs to *Physical Astronomy*, and requires the consi-
deration of the distances and magnitudes as well as of the direc-
tions of the bodies composing the system.

Spherical Astronomy treats specially of the *directions* of the
heavenly bodies; and in this branch, therefore, these bodies are
at any given instant regarded as situated upon the surface of a
sphere of an indefinite radius described about an assumed
centre.   It embraces, therefore, not only the problems which arise
from the diurnal motion, but also such as arise from the annual
motion so far as this affects the apparent positions of the hea-
venly bodies upon the celestial sphere, or their directions from
the assumed centre.

SPHERICAL CO-ORDINATES.

2. The direction of a point may be expressed by the angles
which a line drawn to it from the centre of the sphere, or point
of observation, makes with certain fixed lines of reference.   But,
since such angles are directly measured by arcs on the surface
of the sphere, the simplest method is to assign the position in
which the point appears when projected upon the surface of the
sphere.   For this purpose, a great circle of the sphere, supposed
to be given in position, is assumed as a *primitive circle* of refer-
ence, and all points of the surface are referred to this circle by a
system of *secondaries* or great circles perpendicular to the primi-
tive and, consequently, passing through its poles.   The position
of a point on the surface will then be expressed by two *spherical
co-ordinates:* namely, 1st, the distance of the point from the pri-
mitive circle, measured on a secondary; 2d, the distance inter-
cepted on the primitive between this secondary and some given
point of the primitive assumed as the *origin* of co-ordinates.

We shall have different systems of co-ordinates, according to
the circle adopted as a primitive circle and the point assumed as
the origin.

3. *First system of co-ordinates.—Altitude and azimuth.*—In this
system, the primitive circle is the *horizon*, which is that great
circle of the sphere whose plane touches the surface of the

The *altitude* of a point of the celestial sphere is its distance from the horizon measured on a vertical circle, and its *azimuth* is the arc of the horizon intercepted between this vertical circle and any point of the horizon assumed as an origin. The origin from which azimuths are reckoned is arbitrary; so also is the direction in which they are reckoned; but astronomers usually take the south point of the horizon as the origin, and reckon towards the right hand, from 0° to 360°; that is, completely around the horizon in the direction expressed by writing the cardinal points of the horizon in the order S. W. N. E. We may, therefore, also define azimuth as the angle which the vertical plane makes with the plane of the meridian.

Navigators, however, usually reckon the azimuth from the north or south points, according as they are in north or south latitude, and towards the east or west, according as the point of the sphere considered is east or west of the meridian: so that the azimuth never exceeds 180°. Thus, an azimuth which is expressed according to the first method simply by 200° would be expressed by a navigator in north latitude by N. 20° E., and by a navigator in south latitude by S. 160° E., the letter prefixed denoting the origin, and the letter affixed denoting the direction in which the azimuth is reckoned, or whether the point considered is east or west of the meridian.

When the point considered is in the horizon, it is often referred to the east or west points, and its distance from the nearest of these points is called its *amplitude*. Thus, a point in the horizon whose azimuth is 110° is said to have an amplitude of W. 20° N.

Since by the diurnal motion the observer's horizon is made to change its position in the heavens, the co-ordinates, altitude and azimuth, are continually changing. Their values, therefore, will depend not only upon the observer's position on the earth, but upon the time reckoned at his meridian.

Instead of the altitude of a point, we frequently employ its *zenith distance*, which is the arc of the vertical circle between the point and the zenith. The altitude and zenith distance are, therefore, complements of each other.

We shall hereafter denote altitude by h, zenith distance by ζ, azimuth by A. We shall have then

$$\zeta = 90° - h \qquad h = 90° - \zeta$$

The value of ζ for a point below the horizon will be greater than 90°, and the corresponding value of h, found by the formula $h = 90° - \zeta$, will be negative: so that a negative altitude will express the *depression* of a point below the horizon. Thus, a depression of 10° will be expressed by $h = -10°$, or $\zeta = 100°$.

4. *Second system of co-ordinates.—Declination and hour angle.—*In this system, the primitive circle is the *celestial equator*, or that great circle of the sphere whose plane is perpendicular to the axis of the earth and, consequently, coincides with the plane of the terrestrial equator. This circle is also sometimes called the *equinoctial.*

The diurnal motion of the earth does not change the position of the plane of the equator. The axis of the earth produced to the celestial sphere is called the *axis of the heavens:* the points in which it meets the sphere are the north and south poles of the equator, or the poles of the heavens.

Secondaries to the equator are called *circles of declination*, and also *hour circles.* Since the plane of the celestial meridian passes through the axis of the equator, it is also a secondary to the equator, and therefore also a circle of declination.

Parallels of declination are small circles parallel to the equator.

The *declination* of a point of the sphere is its distance from the equator measured on a circle of declination, and its *hour angle* is the angle at either pole between this circle of declination and the meridian. The hour angle is measured by the arc of the equator intercepted between the circle of declination and the meridian. As the meridian and equator intersect in two points, it is necessary to distinguish which of these points is taken as the origin of hour angles, and also to know in what direction the arc which measures the hour angle is reckoned. Astronomers reckon from that point of the equator which is on the meridian above the horizon, towards the west,—that is, in the direction of the apparent diurnal motion of the celestial sphere,—and from 0° to 360°, or from 0^h to 24^h, allowing 15° to each hour.

Of these co-ordinates, the declination is not changed by the diurnal motion, while the hour angle depends only on the time at the meridian of the observer, or (which is the same thing) on the position of his meridian in the celestial sphere. All the observers on the same meridian at the same instant will, for the same star, reckon the same declination and hour angle. We have

thus introduced co-ordinates of which one is wholly independent of the observer's position and the other is independent of his latitude.

We shall denote declination by δ, and north declination will be distinguished by prefixing to its numerical value the sign $+$, and south declination by the sign $-$.

We shall sometimes make use of the *polar distance* of a point, or its distance from one of the poles of the equator. If we denote it by P, the north polar distance will be found by the formula

$$P = 90° - \delta$$

and the south polar distance by the formula

$$P = 90° + \delta$$

The hour angle will generally be denoted by t. It is to be observed that as the hour angle of a celestial body is continually increasing in consequence of the diurnal motion, it may be conceived as having values greater than 360°, or 24ʰ, or greater than any given multiple of 360°. Such an hour angle may be regarded as expressing the time elapsed since some given passage of the body over the meridian. But it is usual, when values greater than 360° result from any calculation, to deduct 360°. Again, since hour angles reckoned towards the west are always positive, hour-angles reckoned towards the east must have the negative sign: so that an hour angle of 300°, or 20ʰ, may also be expressed by —60°, or —4ʰ.

5. *Third system of co-ordinates.—Declination and right ascension.—* In this system, the primitive plane is still the equator, and the first co-ordinate is the same as in the second system, namely, the declination. The second co-ordinate is also measured on the equator, but from an origin which is not affected by the diurnal motion. Any point of the celestial equator might be assumed as the origin; but that which is most naturally indicated is the vernal equinox, to define which some preliminaries are necessary.

The *ecliptic* is the great circle of the celestial sphere in which the sun appears to move in consequence of the earth's motion in its orbit. The position of the ecliptic is not absolutely fixed in space; but, according to the definition just given, its position at any instant coincides with that of the great circle in which the

sun appears to be moving at that instant. Its annual change is, however, very small, and its daily change altogether insensible.

The *obliquity of the ecliptic* is the angle which it makes with the equator.

The points where the ecliptic and equator intersect are called the *equinoctial points*, or the *equinoxes;* and that diameter of the sphere in which their planes intersect is the *line of equinoxes.*

The *vernal equinox* is the point through which the sun *ascends* from the southern to the northern side of the equator; and the *autumnal equinox* is that through which the sun *descends* from the northern to the southern side of the equator.

The *solstitial points*, or *solstices*, are the points of the ecliptic 90° from the equinoxes. They are distinguished as the northern and southern, or the summer and winter solstices.

The *equinoctial colure* is the circle of declination which passes through the equinoxes. The *solstitial colure* is the circle of declination which passes through the solstices. The equinoxes are the poles of the solstitial colure.

By the annual motion of the earth, its axis is carried very nearly parallel to itself, so that the plane of the equator, which is always at right angles to the axis, is very nearly a fixed plane of the celestial sphere. The axis is, however, subject to small changes of direction, the effect of which is to change the position of the intersection of the equator and the ecliptic, and hence, also, the position of the equinoxes. In expressing the positions of stars, referred to the vernal equinox, at any given instant, the actual position of the equinox at the instant is understood, unless otherwise stated.

The *right ascension* of a point of the sphere is the arc of the equator intercepted between its circle of declination and the vernal equinox, and is reckoned from the vernal equinox *eastward* from 0° to 360°, or, in time, from 0^h to 24^h.

The point of observation being supposed at the centre of the earth, neither the declination nor the right ascension will be affected by the diurnal motion: so that these co-ordinates are wholly independent of the observer's position on the surface of the earth. Their values, therefore, vary only with the time, and are given in the ephemerides as functions of the time reckoned at some assumed meridian.

We shall generally denote right ascension by α. As its value reckoned towards the east is positive, a negative value resulting

done by referring his zenith to the primitive circle in the same manner as in the case of any other point.

In the first system, the primitive circle being the horizon, of which the zenith is the pole, the altitude of the zenith is always 90°, and its azimuth is indeterminate.

In the second system, the declination of the zenith is the same as the terrestrial latitude of the observer, and its hour angle is zero. The declination of the zenith of a place is called the *geographical latitude*, or simply the latitude, and will be hereafter denoted by φ. North latitudes will have the sign $+$; south latitudes, the sign $-$.

In the third system, the declination of the zenith is, as before, the latitude of the observer, and its right ascension is the same as the hour angle of the vernal equinox.

In the fourth system, the celestial latitude of the zenith is the same as the zenith distance of the nonagesimal, and its celestial· longitude is the longitude of the nonagesimal.

It is evident, from the definitions which have been given, that the problem of determining the latitude of a place by astro-nomical observation is the same as that of determining the declination of the zenith; and the problem of finding the lon-gitude may be resolved into that of determining the right ascension of the meridian at a time when that of the prime meridian is also given, since the longitude is the arc of the equator intercepted between the two meridians, and is, conse-quently, the difference of their right ascensions.

8. The preceding definitions are exemplified in the following figures.

Fig. 1 is a stereographic projection of the sphere upon the plane of the horizon, the projecting point being the nadir. Since the planes of the equator and horizon are both perpendicular to that of the meridian, their intersection is also perpendicular to it; and hence the equator WQE passes through the east and west points of the horizon. All vertical circles passing

Fig. 1.

through the projecting point will be projected into straight lines, as the meridian NZS, the prime vertical WZE, and the vertical circle ZOH drawn through any point O of the·surface
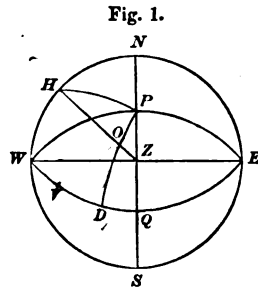
Hence the latitude of the observer is always equal to the altitude of the north pole. For an observer in south latitude, the north pole is below the horizon, and its altitude is a negative quantity: so that the definition of latitude as the altitude of the north pole is perfectly general if we give south latitudes the negative sign. The south latitude of an observer considered independently of its sign is equal to the altitude of the south pole above his horizon, the elevation of one pole being always equal to the depression of the other.

9. *Numerical expression of hour angles.*—The equator, upon which hour angles are measured, may be conceived to be divided into 24 equal parts, each of which is the measure of one hour, and is equivalent to $\frac{1}{24}$ of 360°, or to 15°. The hour is divided sexagesimally into minutes and seconds of *time*, distinguished from minutes and seconds of *arc* by the letters ᵐ and ˢ instead of the accents ′ and ″. We shall have, then,

$$1^h = 15° \qquad 1^m = 15′ \qquad 1^s = 15″$$

To convert an angle expressed in time into its equivalent in arc, multiply by 15 and change the denominations ʰ ᵐ ˢ into ° ′ ″; and to convert arc into time, divide by 15 and change ° ′ ″ into ʰ ᵐ ˢ. The expert computer will readily find ways to abridge these operations in practice. It is well to observe, for this purpose, that from the above equalities we also have,

$$1° = 4^m \qquad 1′ = 4^s$$

and that we may therefore convert degrees and minutes of arc into time by multiplying by 4 and changing ° ′ into ᵐ ˢ; and reciprocally.

TRANSFORMATION OF SPHERICAL CO-ORDINATES.

10. *Given the altitude* (h) *and azimuth* (A) *of a star, or of any point of the sphere, and the latitude* (φ) *of the observer, to find the declination* (δ) *and hour angle* (t) *of the star or the point.* In other words, to transform the co-ordinates of the first system into those of the second.

This problem is solved by a direct application of the formulæ of Spherical Trigonometry to the triangle *POZ*, Fig. 1, in which, *O* being the given star or point, we have three parts given,

to determine such arcs and angles by both the sine and the cosine, in order to fix the quadrant in which their values are to be taken. It has been shown in Spherical Trigonometry that when we consider the *general* triangle, or that in which values are admitted greater than 180°, there are two solutions of the triangle in every case, but that the ambiguity is removed and one of these solutions excluded " when, in addition to the other data, the sign of the sine or cosine of one of the required parts is given." [Sph. Trig. Art. 113.] In our present problem the sign of cos δ is given, since it is necessarily positive; for δ is always numerically less than 90°, that is, between the limits +90° and —90°. Hence cos t has the sign of the second member of (2) or (5), and sin t the sign of the second member of (3) or (6), and t is to be taken in the quadrant required by these two signs. Since h also falls between the limits +90° and —90°, or ζ between 0° and 180°, cos h, or sin ζ, is positive, and therefore by (3) or (6) sin t has the sign of sin A; that is, when $A < 180°$ we have $t < 180°$, and when $A > 180°$ we have $t > 180°$,—conditions which also follow directly from the nature of our problem, since the star is west or east of the meridian according as $A < 180°$ or $A > 180°$. The formula (1) or (4) fully determines δ, which will always be taken less than 90°, positive or negative according to the sign of its sine.*

To adapt the equations (4), (5), and (6) for logarithmic computation, let m and M be assumed to satisfy the conditions [Pl. Trig. Art. 174],

$$\left.\begin{array}{l} m \sin M = \sin \zeta \cos A \\ m \cos M = \cos \zeta \end{array}\right\} \quad (7)$$

the three equations may then be written as follows:

$$\left.\begin{array}{l} \sin \delta = m \sin (\varphi - M) \\ \cos \delta \cos t = m \cos (\varphi - M) \\ \cos \delta \sin t = \sin \zeta \sin A \end{array}\right\} \quad (8)$$

If we eliminate m from these equations, the solution takes the following convenient form:

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* There are, however, special problems in which it is convenient to depart from this general method, and to admit declinations greater than 90°, as will be seen hereafter.

$$\left.\begin{array}{l} \tan M = \tan \zeta \cos A \\ \tan t \;= \dfrac{\tan A \sin M}{\cos (\varphi - M)} \\ \tan \delta \;= \tan (\varphi - M) \cos t \end{array}\right\} \quad (9)$$

in the use of which, we must observe to take t greater or less than 180° according as A is greater or less than 180°, since the hour angle and the azimuth must fall on the same side of the meridian.

EXAMPLE.—In the latitude $\varphi = 38° \, 58' \, 53''$, there are given for a certain star $\zeta = 69° \, 42' \, 30''$, $A = 300° \, 10' \, 30''$; required δ and t. The computation by (9) may be arranged as follows :*

		log tan ζ	0.4820966		
$\phi =$	38° 58′ 53″	log cos A	9.7012595	log tan A	n0.2855026
$M =$	53 39 41.98	log tan M	0.1333561	log sin M	9.9060828
$\phi - M =$	-14 40 48.98	log tan $(\phi - M)$	n9.4182633	log sec $(\phi - M)$	0.0144141
$t =$	304 55 26.49	log cos t	9.7577677	log tan t	n0.1559995
$\delta =$	-8 31 46.56	log tan δ	n9.1760310		

Converting the hour angle into time, we have $t = 20^{\mathrm{h}} \, 19^{\mathrm{m}} \, 41^{\mathrm{s}}.766$.

11. The angle POZ, Fig. 1, between the vertical circle and the declination circle of a star, is frequently called the *parallactic angle*, and will here be denoted by q. To find its value from the data ζ, A, and φ, we have the equations

$$\left.\begin{array}{l} \cos \delta \cos q = \sin \zeta \sin \varphi + \cos \zeta \cos \varphi \cos A \\ \cos \delta \sin q = \qquad\qquad\qquad \cos \varphi \sin A \end{array}\right\} \quad (10)$$

which may be solved in the following form :

$$\left.\begin{array}{l} f \sin F = \sin \zeta \\ f \cos F = \cos \zeta \cos A \\ \cos \delta \cos q \;= f \cos (\varphi - F) \\ \cos \delta \sin q \;= \cos \varphi \sin A \end{array}\right\} \quad (11)$$

or in the following :

$$\left.\begin{array}{l} g \sin G = \sin \varphi \\ g \cos G = \cos \varphi \cos A \\ \cos \delta \cos q = g \cos (\zeta - G) \\ \cos \delta \sin q = \cos \varphi \sin A \end{array}\right\} \quad (12)$$

or again in the following :

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* In this work the letter n prefixed to a logarithm indicates that the number to which it corresponds is to have the negative sign.

$$\left.\begin{array}{l} \tan G = \tan \varphi \sec A \\[1mm] \tan q = \dfrac{\tan A \cos G}{\cos (\zeta - G)} \end{array}\right\} \quad (13)$$

and, in the use of the last form, it is to be observed that q is to be taken greater or less than 180° according as A is greater or less than 180°, as is evident from the preceding forms.

12. If, in a given latitude, the azimuth of a star of known declination is given, its hour angle and zenith distance may be found as follows. We have

$$\cos t \sin \varphi - \sin t \cot A = \cos \varphi \tan \delta$$
$$\cos \zeta \sin \varphi - \sin \zeta \cos \varphi \cos A = \sin \delta$$

The solution of the first of these is effected by the equations

$$b \sin B = \sin \varphi$$
$$b \cos B = \cot A$$
$$\sin (B - t) = \frac{\cos \varphi \tan \delta}{b}$$

and that of the second by

$$c \sin C = \sin \varphi$$
$$c \cos C = \cos \varphi \cos A$$
$$\sin (C - \zeta) = \frac{\sin \delta}{c}$$

13. Finally, if from the given altitude and azimuth we wish to find the declination, hour angle, and parallactic angle at the same time, it will be convenient to use Gauss's Equations, which for the triangle ABC, Fig. 3, are

$$\left.\begin{array}{l} \cos \tfrac{1}{2} a \sin \tfrac{1}{2} (B + C) = \cos \tfrac{1}{2} (b - c) \cos \tfrac{1}{2} A \\ \cos \tfrac{1}{2} a \cos \tfrac{1}{2} (B + C) = \cos \tfrac{1}{2} (b + c) \sin \tfrac{1}{2} A \\ \sin \tfrac{1}{2} a \sin \tfrac{1}{2} (B - C) = \sin \tfrac{1}{2} (b - c) \cos \tfrac{1}{2} A \\ \sin \tfrac{1}{2} a \cos \tfrac{1}{2} (B - C) = \sin \tfrac{1}{2} (b + c) \sin \tfrac{1}{2} A \end{array}\right\} \quad (B)$$

which are to be solved in the usual manner [Sph. Trig. Art. 116] after substituting the values $A = 180° - A$, $b = h$, $c = 90° - \varphi$, $a = 90° - \delta$, $B = t$, $C = q$.

14. *Given the declination* (δ) *and hour angle* (t) *of a star, and the latitude* (φ), *to find the zenith distance* (ζ) *and azimuth* (A) *of the star.* That is, to transform the co-ordinates of the second system into those of the first.

We take the same general equations (A) of Spherical Trigonometry which have been employed in the solution of the pre-

For verification we can use the equation

$$\sin \zeta \sin A = \cos \delta \sin t$$

log sin ζ	9.9721748	log cos δ	9.9951697
log sin A	9.9367621	log sin t	9.9137672
	9.9089369		9.9089369

EXAMPLE 2.—In latitude $\varphi = -48°\ 32'$, there are given for a star, $\delta = 44°\ 6'\ 0''$, $t = 17^h\ 25^m\ 4^s$; required A and ζ.

We find $A = 241°\ 53'\ 33''.2$, $\zeta = 126°\ 25'\ 6''.6$; the star is below the horizon, and its negative altitude, or depression, is $h = -36°\ 25'\ 6''.6$.

If the zenith distance of the same star is to be frequently computed on the same night at a given place, it will be most readily done by the following method. In the first equation of (14) substitute

$$\cos t = 1 - 2 \sin^2 \tfrac{1}{2} t$$

then we have

$$\cos \zeta = \cos (\varphi \smallsmile \delta) - 2 \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t$$

where $\varphi \smallsmile \delta$ signifies either $\varphi - \delta$ or $\delta - \varphi$, and if $\delta > \varphi$ the latter form is to be used. Subtracting both members from unity, we obtain

$$\sin^2 \tfrac{1}{2} \zeta = \sin^2 \tfrac{1}{2} (\varphi \smallsmile \delta) + \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t$$

Now let

$$m = \sqrt{\cos \varphi \cos \delta}$$
$$n = \sin \tfrac{1}{2} (\varphi \smallsmile \delta)$$

then we have

$$\sin \tfrac{1}{2} \zeta = n \sqrt{1 + \frac{m^2 \sin^2 \tfrac{1}{2} t}{n^2}}$$

and hence, by taking an auxiliary N such that

$$\tan N = \frac{m}{n} \sin \tfrac{1}{2} t$$

we have

$$\sin \tfrac{1}{2} \zeta = \frac{n}{\cos N} = \frac{m}{\sin N} \sin \tfrac{1}{2} t$$

$$\left. \right\} \quad (17)$$

The second form for $\sin \tfrac{1}{2} \zeta$ will be more precise than the first when $\sin N$ is greater than $\cos N$.

The quantities m and n will be constant so long as the declination does not vary.

15. If the parallactic angle q (Art. 11) and the zenith distance

the point F changes its position on the horizon with the time; but its position depends *only* on the time or the hour angle ZPO, and not upon the declination of O. The elements of the position of F may therefore be previously computed for successive values of t.

We have in the triangle PFS, right-angled at S, $FPS = t$, $PS = 180° - \varphi$; and if we put

$$\mathfrak{A} = FS, \quad B = PF - 90°, \quad \gamma = 180° - PFS.$$

we find

$$\tan \mathfrak{A} = \sin \varphi \tan t, \quad \tan B = \cot \varphi \cos t, \quad \cot \gamma = \sin B \tan t$$

We have now in the triangle HOF, right-angled at H,

$$B + \delta = OF, \quad \gamma = HFO, \quad h = OH,$$

and if we put

$$u = HF = HS - FS = A - \mathfrak{A},$$

we find

$$\tan u = \cos \gamma \tan (B + \delta) \qquad A = \mathfrak{A} + u$$
$$\sin h = \sin \gamma \sin (B + \delta) \qquad \text{or, } \tan h = \tan \gamma \sin u.$$

To find the parallactic angle $q = POZ$, we have in the triangle HOF

$$\tan q = \cot \gamma \sec (B + \delta)$$

In the Gaussian table for Altona as given in the "Hülfstafeln" we find five columns, which give for the argument t, the quantities $\mathfrak{A}$, B, log cos γ, log sin γ, log cot γ, the last three under the names log C, log D, and log E, respectively. With the aid of this table, then, the labor of finding any one of the quantities h, A, q is reduced to the addition of two logarithms, namely:

$$\tan u = C \tan (B + \delta) \qquad \sin h = D \sin (B + \delta)$$
$$A = \mathfrak{A} + u \qquad \tan q = E \sec (B + \delta)$$

The formulæ for the inverse problem (of Art. 10) may also be found thus. Let G be the intersection of the equator and the vertical circle through O, and put $B = HG$, $u = DG$, $\mathfrak{A} = QG$, $\gamma = ZGQ$; then we readily find

$$\tan \mathfrak{A} = \sin \varphi \tan A, \quad \tan B = \cot \varphi \cos A, \quad \cot \gamma = \sin B \tan A$$

which are of the same form as those given above, with the exchange of A for t. Hence the same table gives also the elements of the point G, by entering with the argument "azimuth," expressed in time, instead of the hour angle. We then have $t =$

If δ is but little less than φ, the star will be near the zenith, and, as in the preceding article, we shall obtain a more accurate solution as follows:

Put
$$k = \sqrt{[\sin(\varphi + \delta)\sin(\varphi - \delta)]}$$
then
$$\sin t = \frac{k}{\sin\varphi\cos\delta} \qquad \sin\zeta = \frac{k}{\sin\varphi} \qquad \cos q = \frac{k}{\cos\delta} \qquad (26)$$

We may also deduce the following convenient and accurate formulæ for the case where the star's declination is nearly equal to the latitude [see Sph. Trig. Arts. 60, 61, 62]:

$$\left.\begin{array}{c} \tan\tfrac{1}{2}t = \sqrt{\left(\dfrac{\sin(\varphi - \delta)}{\sin(\varphi + \delta)}\right)} \\[2mm] \tan\tfrac{1}{2}\zeta = \sqrt{\left(\dfrac{\tan\tfrac{1}{2}(\varphi - \delta)}{\tan\tfrac{1}{2}(\varphi + \delta)}\right)} \\[2mm] \tan(45° - \tfrac{1}{2}q) = \sqrt{[\tan\tfrac{1}{2}(\varphi + \delta)\tan\tfrac{1}{2}(\varphi - \delta)]} \end{array}\right\} \quad (27)$$

If $\delta > \varphi$, these values become imaginary; that is, the star cannot cross the prime vertical.

EXAMPLE.—Required the hour angle and zenith distance of the star 12 *Canum Venaticorum* ($\delta = + 39° 5' 20''$) when on the prime vertical of Cincinnati ($\varphi = + 39° 5' 54''$).

$\varphi - \delta =$ 0° 0′ 34″	$\tfrac{1}{2}(\varphi - \delta) =$ 0° 0′ 17″
$\varphi + \delta =$ 78 11 14	$\tfrac{1}{2}(\varphi - \delta) =$ 39 5 37
log sin $(\varphi - \delta)$ 6.21705	log tan $\tfrac{1}{2}(\varphi - \delta)$ 5.91602
log sin $(\varphi + \delta)$ 9.99070	log tan $\tfrac{1}{2}(\varphi + \delta)$ 9.90982
2)6.22635	2)6.00620
log tan $\tfrac{1}{2}t$ 8.11318	log tan $\tfrac{1}{2}\zeta$ 8.00310
$\tfrac{1}{2}t = 0° 44' 36''.6$	$\tfrac{1}{2}\zeta = 0° 34' 37''.3$
$t = 1° 29' 13''.2$	$\zeta = 1° 9' 14''.6$
$= 0^h 5^m 56^s.88$	

20. *To find the amplitude and hour angle of a given star when in the horizon.*—If the star is at H, Fig. 1, we have in the triangle *PHN*, right-angled at N, $PN = \varphi$, $HPN = 180° - t$, $PH = 90° - \delta$; and if the amplitude WH is denoted by a, we have $HN = 90° - a$. This triangle gives, therefore,

$$\left.\begin{array}{l} \sin a = \sec\varphi\sin\delta \\ \cos t = -\tan\varphi\tan\delta \end{array}\right\} \quad (28)$$

21. *Given the hour angle (t) of a star, to find its right ascension (α).* —Transformation from our second system of co-ordinates to the third.

There must evidently be given also the position of the meridian with reference to the origin of right ascensions. Suppose then in Fig. 1 we know the right ascension of the meridian, or $VQ = \Theta$, then we have $VD = VQ - DQ$, that is,

$$a = \Theta - t$$

Conversely, if a and Θ are known, we have

$$t = \Theta - a$$

The methods of finding Θ at a given time will be considered hereafter.

22. *Given the zenith distance of a known star at a given place, to find the star's hour angle, azimuth, and parallactic angle.*

In this case there are given in the triangle POZ, Fig. 1, the three sides $ZO = \zeta$, $POP = 90° - \delta$, $PZ = 90° - \varphi$, to find the angles $ZPO = t$, $PZO = 180° - A$, and $POZ = q$. The formula for computing an angle B of a spherical triangle ABC, whose sides are a, b, c, is either

$$\sin \tfrac{1}{2} B = \sqrt{\left(\frac{\sin (s - a) \sin (s - c)}{\sin a \sin c} \right)}$$

$$\cos \tfrac{1}{2} B = \sqrt{\left(\frac{\sin s \sin (s - b)}{\sin a \sin c} \right)}$$

or $\quad \tan \tfrac{1}{2} B = \sqrt{\left(\frac{\sin (s - a) \sin (s - c)}{\sin s \sin (s - b)} \right)}$

in which $s = \tfrac{1}{2} (a + b + c)$. We have then only to suppose B to represent one of the angles of our astronomical triangle, and to substitute the above corresponding values of the sides, to obtain the required solution.

This substitution will be carried out hereafter in those cases where the problem is practically applied.

23. *Given the declination (δ) and the right ascension (α) of a star, and the obliquity of the ecliptic (ε), to find the latitude (β) and the longitude (λ) of the star.*—Transformation from the third system of co-ordinates to the fourth.

The solution of this problem is similar to that of Art. 10.

The analogy of the two will be more apparent if we here repre-
sent the sphere projected on the plane of
the equator as in Fig. 6, where $VBUC$ is
the equator, P its pole; VAU the ecliptic,
P' its pole, and consequently $CP'PB$ the
solstitial colure; POD, $P'OL$, circles of
declination and latitude drawn through the
star O. Since the angle which two great
circles make with each other is equal to
the angular distance of their poles, we have
$PP' = \epsilon$; and since the angle $P'PO$ is
measured by CD and $PP'O$ by AL, we have in the triangle
$PP'O$

Fig. 6.

$$
\begin{array}{ccccc}
P'PO, & PP'O, & P'O, & PO, & PP' \\
90° + a, & 90° - \lambda, & 90° - \beta, & 90° - \delta, & \epsilon
\end{array}
$$

which, substituted respectively for

$$
\begin{array}{ccccc}
A, & B & a, & b, & c,
\end{array}
$$

in the general equations (𝔄), Art. 10, give

$$
\left.
\begin{aligned}
\sin \beta &= \cos \epsilon \sin \delta - \sin \epsilon \cos \delta \sin a \\
\cos \beta \sin \lambda &= \sin \epsilon \sin \delta + \cos \epsilon \cos \delta \sin a \\
\cos \beta \cos \lambda &= \qquad\qquad \cos \delta \cos a
\end{aligned}
\right\} \quad (29)
$$

which are the required formulæ of transformation. Adapting
for logarithmic computation, we have

$$
\left.
\begin{aligned}
m \sin M &= \sin \delta \\
m \cos M &= \cos \delta \sin a \\
\sin \beta &= m \sin (M - \epsilon) \\
\cos \beta \sin \lambda &= m \cos (M - \epsilon) \\
\cos \beta \cos \lambda &= \cos \delta \cos a
\end{aligned}
\right\} \quad (30)
$$

in which m is a positive number.

A still more convenient form is obtained by substituting

$$
k = \frac{m}{\cos \delta} \qquad\qquad k' = \frac{\cos \beta}{m}
$$

by which we find

$$
\left.
\begin{aligned}
k \sin M &= \tan \delta \\
k \cos M &= \sin a \\
k' \sin \lambda &= \cos (M - \epsilon) \\
k' \cos \lambda &= \cos M \cot a \\
\tan \beta &= \sin \lambda \tan (M - \epsilon)
\end{aligned}
\right\} \quad (31)
$$

For verification : $\quad \dfrac{\cos \beta \sin \lambda}{\cos \delta \sin a} = \dfrac{\cos (M - \epsilon)}{\cos M}$

EXAMPLE.—Given δ, a, and ϵ as below, to find β and λ. Computation by (31).

$$\delta = -16° 22' 35''.45 \qquad \log \sin \lambda \qquad n8.0897286$$
$$a = \quad 6 \quad 33 \quad 29 .30 \qquad \log \tan (M - \epsilon) \; 1.4114658$$
$$\epsilon = \quad 23 \quad 27 \quad 31 .72 \qquad \overline{\log \tan \beta \qquad n9.5011944}$$
$$\log \tan \delta = \log k \sin M \; n9.4681562 \qquad \beta = -17° 35' 37''.51$$
$$\log \sin a = \log k \cos M \quad 9.0577093$$
$$M = -68° 45' 41''.87 \qquad Verification.$$
$$M - \epsilon = -92 \quad 13 \quad 13 .59 \qquad \log \cos \beta \sin \lambda \; n8.0689234$$
$$\log \cos \delta \sin a \quad 9.0397224$$
$$\log \cos M \quad 9.5590070 \qquad \log \dfrac{\cos (M - \epsilon)}{\cos M} \; n9.0292010$$
$$\log \cot a \quad 0.9394396$$
$$\overline{\log k' \cos \lambda \quad 0.4984466}$$
$$\log \cos (M - \epsilon) = \log k' \sin \lambda \; n8.5882080$$
$$\lambda = 359° 17' 43''.91$$

Tables for facilitating the above transformation, based upon the same method as that employed in Art. 16, are given in the American Ephemeris and Berlin Jahrbuch. The formulæ there used may be obtained from Fig. 6, in which the points F and G are used precisely as in Fig. 4 of Art. 16.

24. If we denote the angle at the star, or $P'OP$, by $90° - E$, the solution of the preceding problem by Gauss's Equations is

$$
\left.
\begin{aligned}
\cos (45° - \tfrac{1}{2} \beta) \sin \tfrac{1}{2} (E + \lambda) &= \sin [45° - \tfrac{1}{2} (\epsilon - \delta)] \sin (45° + \tfrac{1}{2} a) \\
\cos (45° - \tfrac{1}{2} \beta) \cos \tfrac{1}{2} (E + \lambda) &= \cos [45° - \tfrac{1}{2} (\epsilon + \delta)] \cos (45° + \tfrac{1}{2} a) \\
\sin (45° - \tfrac{1}{2} \beta) \sin \tfrac{1}{2} (E - \lambda) &= \sin [45° - \tfrac{1}{2} (\epsilon + \delta)] \cos (45° + \tfrac{1}{2} a) \\
\sin (45° - \tfrac{1}{2} \beta) \cos \tfrac{1}{2} (E - \lambda) &= \cos [45° - \tfrac{1}{2} (\epsilon - \delta)] \sin (45° + \tfrac{1}{2} a)
\end{aligned}
\right\} (32)
$$

25. If the angle at the star is required when the Gaussian Equations have not been employed, we have from the triangle POP', Fig. 6, putting $P'OP = \eta = 90° - E$,

$$\cos \beta \cos \eta = \cos \epsilon \cos \delta + \sin \epsilon \sin \delta \sin \alpha$$
$$\cos \beta \sin \eta = \sin \epsilon \cos \alpha$$

or, adapted for logarithms,

$$\left.\begin{array}{l} n \sin N = \sin \epsilon \sin \alpha \\ n \cos N = \cos \epsilon \\ \cos \beta \cos \eta = n \cos (N - \delta) \\ \cos \beta \cos \eta = \sin \epsilon \cos \alpha \end{array}\right\} \quad (33)$$

26. *Given the latitude (β) and longitude (λ) of a star, and the obliquity of the ecliptic (ϵ), to find the declination and right ascension of the star.*

By the process already employed, we derive from the triangle $PP'O$, Fig. 6, for this case,

$$\left.\begin{array}{rl} \sin \delta = & \cos \epsilon \sin \beta + \sin \epsilon \cos \beta \sin \lambda \\ \cos \delta \sin \alpha = & - \sin \epsilon \sin \beta + \cos \epsilon \cos \beta \sin \lambda \\ \cos \delta \cos \alpha = & \cos \beta \cos \lambda \end{array}\right\} \quad (34)$$

which, it will be observed, may be obtained from (29) by interchanging α with λ, and δ with β, and at the same time changing the sign of ϵ, that is, putting $-\epsilon$ for ϵ, and, consequently, $-\sin \epsilon$ for $\sin \epsilon$.

For logarithmic computation, we have

$$\left.\begin{array}{rl} m \sin M = & \sin \beta \\ m \cos M = & \cos \beta \sin \lambda \\ \sin \delta = & m \sin (M + \epsilon) \\ \cos \delta \sin \alpha = & m \cos (M + \epsilon) \\ \cos \delta \cos \alpha = & \cos \beta \cos \lambda \end{array}\right\} \quad (35)$$

or the following, analogous to (31):

$$\left.\begin{array}{rl} k \sin M = & \tan \beta \\ k \cos M = & \sin \lambda \\ k' \sin \alpha = & \cos (M + \epsilon) \\ k' \cos \alpha = & \cos M \cot \lambda \\ \tan \delta = & \sin \alpha \tan (M + \epsilon) \end{array}\right\} \quad (36)$$

For verification : $\quad \dfrac{\cos \delta \sin \alpha}{\cos \beta \sin \lambda} = \dfrac{\cos (M + \epsilon)}{\cos M}$

27. The angle at the star, POP', being denoted, as in Art. 24,

by $90° - E$, the solution of this problem by the Gaussian Equations is

$$\left.\begin{aligned}
\sin (45°- \tfrac{1}{2} \delta) \sin \tfrac{1}{2} (E + a) &= \sin [45°- \tfrac{1}{2} (\varepsilon + \beta)] \sin (45°+ \tfrac{1}{2} \lambda) \\
\sin (45°- \tfrac{1}{2} \delta) \cos \tfrac{1}{2} (E + a) &= \cos [45°- \tfrac{1}{2} (\varepsilon - \beta)] \cos (45°+ \tfrac{1}{2} \lambda) \\
\cos (45°- \tfrac{1}{2} \delta) \sin \tfrac{1}{2} (E - a) &= \cos [45°- \tfrac{1}{2} (\varepsilon - \beta)] \sin (45°+ \tfrac{1}{2} \lambda) \\
\cos (45°- \tfrac{1}{2} \delta) \cos \tfrac{1}{2} (E - a) &= \sin [45°- \tfrac{1}{2} (\varepsilon + \beta)] \cos (45°+ \tfrac{1}{2} \lambda)
\end{aligned}\right\} \quad (37)$$

28. But if the angle $\eta = 90° - E$ is required when the Gaussian Equations have not been employed, we have directly

$$\cos \delta \cos \eta = \cos \varepsilon \cos \beta - \sin \varepsilon \sin \beta \sin \lambda$$
$$\cos \delta \sin \eta = \sin \varepsilon \cos \lambda$$

or, adapted for logarithms,

$$\left.\begin{aligned}
n \sin N &= \sin \varepsilon \sin \lambda \\
n \cos N &= \cos \varepsilon \\
\cos \delta \cos \eta &= n \cos (N + \beta) \\
\cos \delta \sin \eta &= \sin \varepsilon \cos \lambda
\end{aligned}\right\} \quad (38)$$

29. *For the sun*, we may, except when extreme precision is desired, put $\beta = 0$, and the preceding formulæ then assume very simple forms. Thus, if in (34) we put $\sin \beta = 0$, $\cos \beta = 1$, we find

$$\sin \delta = \sin \varepsilon \sin \lambda$$
$$\cos \delta \sin a = \cos \varepsilon \sin \lambda$$
$$\cos \delta \cos a = \cos \lambda$$

whence if any two of the four quantities δ, a, λ, ε are given, we can deduce the other two.

RECTANGULAR CO-ORDINATES.

30. By means of spherical co-ordinates we have expressed only a star's *direction*. To define its position in space completely, another element is necessary, namely, its *distance*. In Spherical Astronomy we consider this element of distance only so far as may be necessary in determining the changes of apparent direction of a star resulting from a change in the point from which it is viewed. For this purpose the rectangular co-ordinates of analytical geometry may be employed.

Three planes of reference are taken at right angles to each other, their common intersection, or origin, being the point of

For, let AB, Fig. 7, be the given great circle orthographically projected upon a plane passing through its axis OP and the given point C; P its pole. The distance of the point C from the plane of the great circle is the perpendicular CD; CE is its distance from the axis; CO its distance from the centre of the sphere; and the angle COP the angular distance from the pole. The truth of the Lemma is, therefore, obvious from the figure.

Fig. 7.
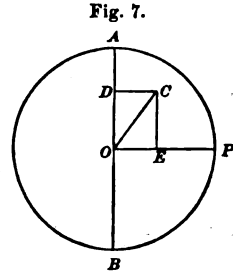
32. The values of the rectangular co-ordinates in our several systems may be found as follows:

First system.—Altitude and azimuth.—Let the primitive plane, or that of the horizon, be the plane of xy; that of the meridian, the plane of xz; that of the prime vertical, the plane of yz. The meridian line is then the axis of x; the east and west line, the axis of y; and the vertical line, the axis of z. Positive x will be reckoned from the origin towards the south, positive y towards the west, and positive z towards the zenith. The *first angle*, or angle of positive values, is therefore the southwest quarter of the hemisphere above the plane of the horizon. Let Z, Fig. 8, be the zenith, S the south point, W the west point of the horizon. These points are respectively the poles of the three great circles of reference; if, then, A is the position of a star on the surface of the sphere as seen from the centre of the earth, and if we put

Fig. 8.
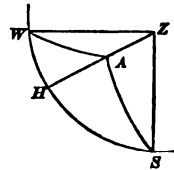
$h =$ altitude of the star $= AH$,
$A =$ azimuth " $= SH$,
$\varDelta =$ its distance from the centre of the sphere

we have immediately, by the preceding Lemma,

$$x = \varDelta \cos AS, \qquad y = \varDelta \cos AW, \qquad z = \varDelta \cos AZ,$$

which, by considering the right triangles AHS, AHW, become

$$\left. \begin{array}{l} x = \varDelta \cos h \cos A \\ y = \varDelta \cos h \sin A \\ z = \varDelta \sin h \end{array} \right\} \quad (39)$$

These equations determine the rectangular co-ordinates x, y, z,

The axis of x is the intersection of the planes of the equator and equinoctial colure, positive towards the vernal equinox; the axis of y is the intersection of the planes of the equator and solstitial colure, positive towards that point whose right ascension is $+90°$; and the axis of z is the axis of the equator, positive towards the north. If then, in Fig. 8, Z is the north pole, W the vernal equinox, A a star in the first angle, projected upon the celestial sphere, and we put

$$\delta = \text{declination of the star} = AH,$$
$$a = \text{right ascension} \quad \text{``} \quad = WH,$$
$$\varDelta = \text{distance from the centre,}$$

while x'', y'', z'' denote the rectangular co-ordinates, we have

$$x'' = \varDelta \cos AW, \qquad y'' = \varDelta \cos AS, \qquad z'' = \varDelta \cos AZ,$$

which become

$$\left. \begin{aligned} x'' &= \varDelta \cos \delta \cos a \\ y'' &= \varDelta \cos \delta \sin a \\ z'' &= \varDelta \sin \delta \end{aligned} \right\} \quad (41)$$

Fourth system.—Celestial latitude and longitude.—Let the plane of the ecliptic be the plane of xy; the plane of the circle of latitude passing through the equinoctial points, the plane of xz; the plane of the circle of latitude passing through the solstitial points, the plane of yz. The positive axis of x is here also the straight line from the centre towards the vernal equinox; the positive axis of y is the straight line from the centre towards the north solstitial point, or that whose longitude is $+90°$; and the positive axis of z is the straight line from the centre towards the north pole of the ecliptic.

If then, in Fig. 8, Z now denotes the north pole of the ecliptic, W the vernal equinox, A the star's place on the sphere, and we put

$$\beta = \text{latitude of the star} \quad = AH,$$
$$\lambda = \text{longitude of the star} = WH,$$
$$\varDelta = \text{distance of the star from the centre,}$$

and x''', y''', z''', denote the rectangular co-ordinates for this system, we have

$$\left. \begin{aligned} x''' &= \varDelta \cos \beta \cos \lambda \\ y''' &= \varDelta \cos \beta \sin \lambda \\ z''' &= \varDelta \sin \beta \end{aligned} \right\} \quad (42)$$

TRANSFORMATION OF RECTANGULAR CO-ORDINATES.

33. For the purposes of Spherical Astronomy, only the most simple cases of the general transformations treated of in analytical geometry are necessary. We mostly consider but two cases:

First. Transformation of rectangular co-ordinates to a new origin, without changing the system of spherical co-ordinates.

The general planes of reference which have been used in this chapter may be supposed to be drawn through any point in space without changing their directions, and therefore without changing the great circles of the infinite celestial sphere which represent them. We thus repeat the same *system* of spherical co-ordinates with various origins and different systems of rectangular co-ordinates, the planes of reference, however, remaining always parallel to the planes of the primitive system.

The transformation from one system of rectangular co-ordinates to a parallel system is evidently effected by the formulæ

$$\left. \begin{array}{l} x_1 = x_2 + a \\ y_1 = y_2 + b \\ z_1 = z_2 + c \end{array} \right\} \quad (43)$$

in which x_1, y_1, z_1 are the co-ordinates of a point in the primitive system; x_2, y_2, z_2 the co-ordinates of the same point in the new system; and a, b, c are the co-ordinates of the new origin taken in the first system.

As we have shown how to express the values of x_1, y_1, z_1 and of x_2, y_2, z_2 in terms of the spherical co-ordinates, we have only to substitute these values in the preceding formulæ to obtain the general relations between the spherical co-ordinates corresponding to the two origins. This is, indeed, the most general method of determining the effect of *parallax*, as will appear hereafter.
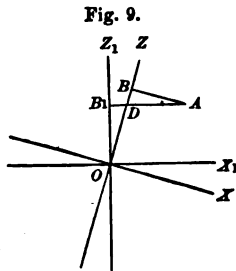
Fig. 9.

Second. Transformation of rectangular co-ordinates when the system of spherical co-ordinates is changed but the origin is unchanged. This amounts to changing the directions of the axes. The cases which occur in practice are chiefly those in which the two systems have one plane in common. Suppose this plane is that of xz, and let OX, OZ, Fig. 9, be the axes of x and z in the first system; OX_1,

$$\cos h \cos A = \sin \varphi \cos \delta \cos t - \cos \varphi \sin \delta$$
$$\cos h \sin A = \cos \delta \sin t$$
$$\sin h = \cos \varphi \cos \delta \cos t + \sin \varphi \sin \delta$$

which agree with (14). We see that when the element of distance is left out of view (as it must necessarily be when the origin is not changed), the transformation by means of rectangular co-ordinates leads to the same forms as the direct application of Spherical Trigonometry. With regard to the entire generality of these formulæ in their application to angles of all possible magnitudes, see Sph. Trig. Chap. IV.

DIFFERENTIAL VARIATIONS OF CO-ORDINATES.

34. It is often necessary in practical astronomy to determine what effect given variations of the data will produce in the quantities computed from them. Where the formulæ of computation are derived directly from a spherical triangle, we can employ for this purpose the equations of *finite differences* [Sph. Trig. Chap. VI.] if we wish to obtain rigorously exact relations, or the simpler differential equations if the variations considered are extremely small. As the latter case is very frequent, I shall deduce here the most useful differential formulæ, assuming as well known the fundamental ones [Sph. Trig. Art. 153],

$$\left.\begin{array}{l} da - \cos C \, db - \cos B \, dc = \sin b \sin C \, dA \\ - \cos C \, da + \quad db - \cos A \, dc = \sin c \sin A \, dB \\ - \cos B \, da - \cos A \, db + \quad dc = \sin a \sin B \, dC \end{array}\right\} \quad (46)$$

From these we obtain the following by eliminating da:

$$\left.\begin{array}{l} \sin C \, db - \cos a \sin B \, dc = \sin b \cos C \, dA + \sin a \, dB \\ - \cos a \sin C \, db + \quad \sin B \, dc = \sin c \cos B \, dA + \sin a \, dC \end{array}\right\} (47)$$

and by eliminating db from these:

$$\sin a \sin B \, dc = \cos b \, dA + \cos a \, dB + dC \qquad (48)$$

If we eliminate dA from (47), we find

$$\cos b \sin C \, db - \cos c \sin B \, dc = \sin c \cos B \, dB - \sin b \sin C \, dC$$

the terms of which being divided either by $\sin b \sin C$, or by its equivalent $\sin c \sin B$, we obtain

$$\cot b \, db - \cot c \, dc = \cot B \, dB - \cot C \, dC \qquad (49)$$

35. As an example, take the spherical triangle formed by the zenith, the pole, and a star, Art. 10, and put

$$A = 180° - A \qquad\qquad a = 90° - \delta$$
$$B = t \qquad\qquad\qquad b = \zeta$$
$$C = q \qquad\qquad\qquad c = 90° - \varphi$$

then the first equations of (46) and (47) give

$$\left.\begin{aligned}
d\delta &= - \cos q \, d\zeta + \sin q \sin \zeta \, dA + \qquad \cos t \, d\varphi \\
\cos \delta \, dt &= \quad \sin q \, d\zeta + \cos q \sin \zeta \, dA + \sin \delta \sin t \, d\varphi
\end{aligned}\right\} \quad (50)$$

which determine the errors $d\delta$ and dt in the values of δ and t computed according to the formulæ (4), (5), and (6), when ζ, A, and φ are affected by the small errors $d\zeta$, dA, and $d\varphi$ respectively.

In a similar manner we obtain

$$\left.\begin{aligned}
d\zeta &= - \cos q \, d\delta + \sin q \cos \delta \, dt + \qquad \cos A \, d\varphi \\
\sin \zeta \, dA &= \quad \sin q \, d\delta + \cos q \cos \delta \, dt - \cos \zeta \sin A \, d\varphi
\end{aligned}\right\} \quad (51)$$

which determine the errors $d\zeta$ and dA in the values of ζ and A computed by (14), when δ, t, and φ are affected by the small errors $d\delta$, dt, and $d\varphi$ respectively.

36. As a second example, take the triangle formed by the pole of the equator, the pole of the ecliptic, and a star, Art. 23. Denoting the angle at the star by η, we find

$$\left.\begin{aligned}
d\beta &= \cos \eta \, d\delta - \sin \eta \cos \delta \, da - \qquad \sin \lambda \, d\varepsilon \\
\cos \beta \, d\lambda &= \sin \eta \, d\delta + \cos \eta \cos \delta \, da + \sin \beta \cos \lambda \, d\varepsilon
\end{aligned}\right\} \quad (52)$$

and reciprocally,

$$\left.\begin{aligned}
d\delta &= \quad \cos \eta \, d\beta + \sin \eta \cos \beta \, d\lambda + \qquad \sin a \, d\varepsilon \\
\cos \delta \, da &= - \sin \eta \, d\beta + \cos \eta \cos \beta \, d\lambda - \sin \delta \cos a \, d\varepsilon
\end{aligned}\right\} \quad (53)$$

CHAPTER II.

TIME—USE OF THE EPHEMERIS—INTERPOLATION—STAR CATALOGUES.

37. TRANSIT.—The instant when any point of the celestial sphere is on the meridian of an observer is designated as the *transit* of that point over the meridian; also the *meridian passage*, and *culmination*. In one complete revolution of the sphere about its axis, every point of it is twice on the meridian, at points which are 180° distant in right ascension. It is therefore necessary to distinguish between the two transits. The meridian is bisected at the poles of the equator: the transit over that half of the meridian which contains the observer's zenith is the *upper* transit, or culmination; that over the half of the meridian which contains the nadir is the *lower* transit, or culmination. At the upper transit of a point its hour angle is zero, or 0^h; at the lower transit, its hour angle is 12^h.

38. The motion of the earth about its axis is perfectly uniform. If, then, the axis of the earth preserved precisely the same direction in space, the apparent diurnal motion of the celestial sphere would also be perfectly uniform, and the intervals between the successive transits of any assumed point of the sphere would be perfectly equal. The effect of changes in the position of the earth's axis upon the transit of stars is most perceptible in the case of stars near the vanishing points of the axis, that is, near the poles of the heavens. We obtain a measure of time *sensibly* uniform by employing the successive transits of a point of the equator. The point most naturally indicated is the *vernal equinox* (also called the First point of Aries, and denoted by the symbol for Aries, ♈).

39. A *sidereal day* is the interval of time between two successive (upper) transits of the true vernal equinox over the same meridian.

The effect of precession and nutation upon the time of transit

of the vernal equinox is so nearly the same at two successive transits, that sidereal days thus defined are *sensibly* equal.   (See Chapter XI. Art. 411.)

The *sidereal time* at any instant is the hour angle of the vernal equinox at that instant, reckoned from the meridian westward from 0^h to 24^h.

When ♈ is on the meridian, the sidereal time is 0^h 0^m 0^s; and this instant is sometimes called *sidereal noon*.

40. A *solar day* is the interval of time between two successive upper transits of the sun over the same meridian.

The *solar time* at any instant is the hour angle of the sun at that instant.

In consequence of the earth's motion about the sun from west to east, the sun appears to have a like motion among the stars, or to be constantly increasing its right ascension; and hence a solar day is longer than a sidereal day.

41. *Apparent and mean solar time.*—If the sun changed its right ascension uniformly, solar days, though not equal to sidereal days, would still be equal to each other.   But the sun's motion in right ascension is not uniform, and this for two reasons:

1st. The sun does not move in the equator, but in the ecliptic, so that, even were the sun's motion in the ecliptic uniform, its equal changes of longitude would not produce equal changes of right ascension; 2d. The sun's motion in the ecliptic is not uniform.

To obtain a uniform measure of time depending on the sun's motion, the following method is adopted.   A fictitious sun, which we shall call the *first mean sun*, is supposed to move uniformly at such a rate as to return to the perigee at the same time with the true sun.   Another fictitious sun, which we shall call the *second mean sun* (and which is often called simply the mean sun), is supposed to move uniformly in the equator at the same rate as the first mean sun in the ecliptic, and to return to the vernal equinox at the same time with it.   Then the time denoted by this second mean sun is perfectly uniform in its increase, and is called *mean time.*

The time which is denoted by the true sun is called the *true* or, more commonly, the *apparent time.*

The instant of transit of the true sun is called *apparent noon,* and the instant of transit of the second mean sun is called *mean noon.*

45. *Time at different meridians.*—The hour angle of the sun at any meridian is called the *local* (solar) time at that meridian.

The hour angle of the sun at the Greenwich meridian at the same instant is the corresponding *Greenwich time.* This time we shall have constant occasion to use, both because longitudes in this country and England are reckoned from the Greenwich meridian, and because the American and British Nautical Almanacs are computed for Greenwich time.*

The difference between the local time at any meridian and the Greenwich time is equal to the longitude of that meridian from Greenwich, expressed in time, observing that $1^h = 15°$.

The difference between the local times of any two meridians is equal to the difference of longitude of those meridians.

In comparing the corresponding times at two different meridians, the most easterly meridian may be distinguished as that at which the time is greatest; that is, latest.

Fig. 10.

If then *PM*, Fig. 10, is any meridian (referred to the celestial sphere), *PG* the Greenwich meridian, *PS* the declination circle through the sun, and if we put

$$T_0 = \text{the Greenwich time} \qquad = GPS,$$
$$T = \text{the local time} \qquad\qquad = MPS,$$
$$L = \text{the west longitude of the meridian } PM = GPM,$$

we have

$$\left. \begin{array}{l} L = T_0 - T \\ T_0 = T + L \end{array} \right\} \quad (54)$$

If the given meridian were east of Greenwich, as *PM'*, we should have its east longitude $= T - T_0$; but we prefer to use the general formula $L = T_0 - T$ in all cases, observing that *east longitudes are to be regarded as negative.*

In the formula (54), T_0 and T are supposed to be reckoned always westward from their respective meridians, and from 0^h to 24^h; that is, T_0 and T are the *astronomical times*, which should, of course, be used in all astronomical computations.

As in almost every computation of practical astronomy we are dependent for some of the data upon the ephemeris,—and these

---

* What we have to say respecting the Greenwich time is, however, equally applicable to the time at any other meridian for which the ephemeris may be computed.

are commonly given for Greenwich,—it is generally the first step in such a computation to deduce an exact or, at least, an approximate value of the Greenwich astronomical time. It need hardly be added that the Greenwich time should never be otherwise expressed than astronomically.*

EXAMPLES.

1. In Longitude 76° 32′ W. the local time is 1856 April 1, 9^h 3^m 10^s A.M.; what is the Greenwich time?

$$
\begin{array}{lrrr}
\text{Local Ast. T. March 31,} & 21^h & 3^m & 10^s \\
\text{Longitude} & + \quad 5 & 6 & 8 \\
\hline
\text{Greenwich T. April 1,} & 2 & 9 & 18 \\
\end{array}
$$

2. In Long. 105° 15′ E. the local time is August 21, 4^h 3^m P.M; what is the Greenwich time?

$$
\begin{array}{lrr}
\text{Local Ast. T. Aug. 21,} & 4^h & 3^m \\
\text{Longitude} & - \quad 7 & 1 \\
\hline
\text{Greenwich T. Aug. 20,} & 21 & 2 \\
\end{array}
$$

3. Long. 175° 30′ W. Loc. T. Sept. 30, 8^h 10^m A.M. $=$ G. T. Sept. 30, 7^h 52^m.

4. Long. 165° 0′ E. Loc. T. Feb. 1, 7^h 11^m P.M. $=$ G. T. Jan. 31, 20^h 11^m.

5. Long. 100° 31′ E. Loc. T. June 1, 0^h M. (Noon) $=$ G. T. May 31, 19^h 42^m.

46. In nautical practice the observer is provided with a chronometer which is regulated to Greenwich time, before sailing, at a place whose longitude is well known. Its error on Greenwich time is carefully determined, as well as its daily gain or loss, that is, its *rate*, so that at any subsequent time the Greenwich time may be known from the indication of the chronometer corrected for its error and the accumulated rate since the date of sailing. As, however, the chronometer has usually only 12^h marked on the dial, it is necessary to distinguish whether it indicates A.M. or P.M. at Greenwich. This is always readily done by means of the observer's *approximate* longitude and local

---

* On this account, chronometers intended for nautical and astronomical purposes should always be marked from 0^h to 24^h, instead of from 0^h to 12^h as is now usual. It is surprising that navigators have not insisted upon this point.

in which E is to be regarded as a positive quantity when it is *additive to apparent time.* The value of E is to be taken from the Nautical Almanac for the Greenwich instant corresponding to the given local time. If apparent time is given, find the Gr. apparent time and take E from page I of the month in the Nautical Almanac; if mean time is given, find the Gr. mean time and take E from page II of the month.

EXAMPLE 1.—In longitude 60° W., 1856 May 24, $3^h 12^m 10^s$ P.M., apparent time; what is the mean time?
We have first

$$
\begin{array}{lrrr}
\text{Local time May 24,} & 3^h & 12^m & 10^s \\
\text{Longitude,} & 4 & 0 & 0 \\
\hline
\text{Gr. app. time May 24,} & 7 & 12 & 10
\end{array}
$$

We must, therefore, find E for the Gr. time, May 24, $7^h 12^m 10^s$, or $7^h.21$. By the Nautical Almanac for 1856, we have E at apparent Greenwich noon May 24 $= -3^m 25^s.43$, and the hourly difference $+ 0^s.224$. Hence at the given time

$$E = -3^m 25^s.43 + 0^s.224 \times 7.21 = -3^m 23^s.81$$

and the required mean time is

$$M = 3^h 12^m 10^s - 3^m 23^s.81 = 3^h 8^m 46^s.19.$$

EXAMPLE 2.—In longitude 60° W., 1856 May 24, $3^h 8^m 46^s.19$ mean time; what is the apparent time?

$$
\begin{array}{llll}
\text{Gr. mean time, May 24, } 7^h 8^m 46^s.19 \ (= 7^h.15) & & \\
E \text{ at mean noon May 24} = - \quad 3^m 25^s.41 & \text{Hourly diff.} = 0^s.224 \\
\text{Correction for } 7^h.15 \quad = \quad + 1.60 & \quad 7.15 \\
\hline
\qquad\qquad\qquad\qquad E = - \quad 3 \quad 23.81 & \quad 1.60
\end{array}
$$

and hence

$$
\begin{array}{rrrr}
M = & 3^h & 8^m & 46^s.19 \\
-E = & + & 3 & 23.81 \\
\hline
A = & 3 & 12 & 10.00
\end{array}
$$

As the equation of time is not a uniformly varying quantity, it is not quite accurate to compute its correction as above, by multiplying the given hourly difference by the number of hours in the Greenwich time, for that process assumes that this hourly difference is the same for each hour. The variations in the hourly difference are, however, so small that it is only when

extreme precision is required that recourse must be had to the more exact method of interpolation which will be given hereafter.

49. *To determine the relative length of the solar and sidereal units of time.*

According to BESSEL, the length of the tropical year (which is the interval between two successive passages of the sun through the mean vernal equinox) is 365.24222 mean solar days;* and since in this time the mean sun has described the whole arc of the equator included between the two positions of the equinox, it has made one transit less over any given meridian than the vernal equinox; so that we have

$$366.24222 \text{ sidereal days} = 365.24222 \text{ mean solar days}$$

whence we deduce

$$1 \text{ sid. day} = \frac{365.24222}{366.24222} \text{ sol. day} = 0.99726957 \text{ sol. day}$$

or

$$24^h \text{ sid. time} = 23^h 56^m 4^s.091 \text{ solar time}$$

Also,

$$1 \text{ sol. day} = \frac{366.24222}{365.24222} \text{ sid. day} = 1.00273791 \text{ sid. day}$$

or

$$24^h \text{ sol. time} = 24^h 3^m 56^s.555 \text{ sid. time}$$

If we put

$$\mu = \frac{366.24222}{365.24222} = 1.00273791$$

and denote by I an interval of mean solar time, by I' the equivalent interval of sidereal time, we always have

$$\left. \begin{array}{l} I' = \mu I = I + (\mu - 1) I = I + .00273791\, I \\ I = \dfrac{I'}{\mu} = I' - (1 - \dfrac{1}{\mu}) I' = I' - .00273043\, I' \end{array} \right\} \quad (55)$$

Tables are given in the Nautical Almanacs to save the labor of computing these equations. In some of these tables, for each solar interval I there is given the equivalent sidereal interval $I' = \mu I$, and reciprocally: in others there are given the correction to be added to I to find I' (*i.e.* the correction .00273791 I),

---

* The length of the tropical year is not absolutely constant. The value given in the text is for the year 1800. Its decrease in 100 years is about 0ˢ.6 (Art. 407).

and the correction to be subtracted from I' to find I (*i.e.* the correction .00273043 I'). The latter form is the most convenient, and is adopted in the American Ephemeris. The correction $(\mu - 1) I$ is frequently called the *acceleration of the fixed stars* (relatively to the sun). The daily acceleration is $3^m 56^s.555$.

50. *To convert the mean solar time at a given meridian into the corresponding sidereal time.*

In Fig. 1, page 25, if PQ is the given meridian, VQ the equator, D the mean sun, V the vernal equinox, and if we put

$$T = DQ = \text{the mean solar time,}$$
$$\Theta = VQ = \text{the sidereal time,}$$
$$= \text{the right ascension of the meridian,}$$
$$V = \text{the right ascension of the mean sun,}$$

we have

$$\Theta = T + V \tag{56}$$

The right ascension of the mean sun, or V, is given in the American Ephemeris, on page II of the month, for each Greenwich mean noon. It is, however, there called the "Sidereal Time," because at mean noon the second mean sun is on the meridian, and its right ascension is also the right ascension of the meridian, or the sidereal time. But this quantity V is uniformly increasing* at the rate of $3^m 56^s.555$ in 24 mean solar hours, or of $9^s.8565$ in one mean hour. To find its value at the given time T, we may first find the Greenwich mean time T_0 by applying the longitude; then, if we put

$$V_0 = \text{the value of } V \text{ at Gr. mean noon,}$$
$$= \text{the "sidereal time" in the ephemeris for the given date,}$$

we have

$$V = V_0 + 9^s.8565 \times T_0$$

in which T_0 must be expressed in hours and decimal parts. It is easily seen that $9^s.8565$ is the acceleration of sidereal time on solar time in one solar hour, and therefore the term $9^s.8565 \times T_0$ is the correction to add to T_0 to reduce it from a solar to a sidereal interval. This term is identical with $(\mu - 1) T_0$ as given by

---

* The sidereal time at mean noon is equal to the *true* R.A. of the mean sun, or it is the R.A. of the mean sun referred to the *true* equinox, and therefore involves the nutation, so that its rate of increase is not, strictly, uniform. But it is sufficiently so for 24 hours to be so regarded in all practical computations. See Chapter XI.

equinox, and is merely the difference between 24^h and the mean sun's right ascension. The hour angle of the mean sun at any instant is then the hour angle of the vernal equinox increased by the value of V' at that instant. To find this value of V', we first reduce the Almanac value to the given meridian by correcting it for the longitude by the table for converting sidereal into mean time; then reduce it to the given sidereal time Θ (which is the elapsed sidereal time since the transit of the vernal equinox over the given meridian) by further correcting it by the same table for this time Θ. We then have the mean time T by the formula

$$T = \Theta + V'$$

It is necessary to observe, however, that if $\Theta + V'$ exceed 24^h it will increase our date by one day; and in that case V' should be taken from the Almanac for a date one day less than the given date; that is, we must in every case take that value which belongs to the Greenwich transit of the vernal equinox immediately *preceding* that over the given meridian.

Example.—Same as in Art. 52.

$$
\begin{aligned}
\Theta &= 19^h\ 41^m\ 57^s.89 \\
\text{May 15, } V_0' &= 20\quad 23\quad\ \ 3\ .88 \\
\text{Corr. for long. } 11^h\ \text{W.} &= \quad -1\quad 48\ .13 \\
\text{Corr. for } 19^h\ 41^m\ 58^s &= \quad -3\quad 13\ .64 \\
\hline
T &= 16\quad\ \ 0\quad\ \ 0\ .00
\end{aligned}
$$

54. *To find the hour angle of a star** at a given time at a given meridian.*

In Fig. 1, we have for the star at O, $DQ = VQ - VD$; that is, if we put

$\Theta =$ the sidereal time,
$a =$ the right ascension of the star,
$t =$ the hour angle   "   "   "

then $t = \Theta - a$ (59)

If α exceeds Θ, this formula will give a negative value of t which will express the hour angle east of the meridian: in that case, if we increase Θ by 24^h before subtracting α, we shall find

---

* We shall use "star," for brevity, to denote any celestial body.

$$
\begin{array}{rl}
& t = \quad 4^h\ 10^m\ 53^s.2 \\
\text{For Aug. 10, } 9^h\ 22^m & a = 16\ \ 29\ \ 26\ .8 \\
& \Theta = \overline{20\ \ 40\ \ 20\ .0} \\
\text{"} \qquad \text{"} & V = \ 9\ \ 18\ \ \ \ 8\ .1 \\
\text{2d approx. value} & T = \overline{11\ \ 22\ \ 11\ .9}
\end{array}
$$

A third approximation, setting out from this value of T, gives us $T = 11^h\ 22^m\ 12^s.32$.

56. The mean time of the meridian passage not only of the moon but of each of the planets is given in the Ephemeris. This quantity is nothing more than the arc of the equator intercepted between the mean sun and the moon's or planet's declination circle. If we denote it by M, we may regard M as the equation between mean time and the *lunar* or *planetary* time, these terms being used instead of "hour angle of the moon" or "hour angle of a planet," just as we use "solar time" to signify "hour angle of the sun." This quantity M is given in the Ephemeris for the instant when the lunar or planetary time is 0^h, and its variation in 1^h of such time is also given in the adjacent column. If, then, when the moon's or a planet's hour angle at a given meridian $= t$, we take out from the Almanac the value of M for the corresponding Greenwich value of t, we shall find the mean time T by simply adding M to t; that is,

$$T = t + M \qquad (61)$$

This is, in fact, the *direct* solution of the problem of the preceding article, and neither requires a previous knowledge of the Greenwich mean time nor introduces the sidereal time. But the Almanac values of M are not given to seconds; and therefore we can use (61) only for making our first approximation to T, after which we proceed as in the last article. The Greenwich value of t with which we take out M is equal to $t + L$, denoting by L the longitude of the given meridian (to be taken with the negative sign when east), and the required value of M is the Almanac value increased by the hourly diff. multiplied by $(t + L)$ in hours. As the hourly diff. of M in the case of the moon is itself variable, we should use that value of it which corresponds to the middle of the interval $t + L$; that is, we should first correct the hourly diff. by the product of *its* hourly change into $\frac{1}{2}(t + L)$ in hours.

EXAMPLE.—Same as Example 3, Art. 55.   We have

$t + L = 2^h\ 10^m\ 53^s.2 = 2^h.18$ $t = 4^h\ 10^m\ 53^s.2$

At Gr. trans. Hour. Diff. $= 2^m.17$ At Gr. trans. Aug. 10, $M = 7\ \ 6\ \ 30$

Variation of H. D. in $1^h 5^m = \underline{\ \ \ .01}$ $2^m.18 \times 2.18 = \underline{+\ \ 4\ \ 45}$

Corrected Hourly Diff. $= 2\ .18$ $T = \overline{11\ \ 22\ \ \ 8\ .2}$

which agrees within 4^s with the true value.   Taking it as a first approximation, and proceeding as in Art. 55, a second approximation gives $T = 11^h\ 22^m\ 12^s.19$.

THE EPHEMERIS, OR NAUTICAL ALMANAC.

57. We have already had occasion to refer to the Ephemeris; but we propose here to treat more particularly of its arrangement and use.

The Astronomical Ephemeris expresses in numbers the actual state of the celestial sphere at given instants of time; that is, it gives for such instants the numerical values of the co-ordinates of the principal celestial bodies, referred to circles whose positions are independent of the diurnal motion of the earth, as declination and right ascension, latitude and longitude; together with the elements of position of the circles of reference themselves.   It also gives the effects of changes of position of the observer 'upon the co-ordinates, or, rather, numbers from which such changes can be readily computed (namely, the parallax, which will be fully considered hereafter), the apparent angular magnitude of the sun, moon, and planets, and, in general, all those phenomena which depend on the time; that is, which may be regarded simply as *functions of the time*.

The *American Ephemeris* is composed of two parts, the first computed for the meridian of Greenwich, in conformity with the *British Nautical Almanac*, especially for the use of navigators; the second computed for the meridian of Washington for the convenience of American astronomers.   The French Ephemeris, *La Connaissance des Temps*, is computed for the meridian of Paris; the German, *Berliner Astronomisches Jahrbuch*, for the meridian of Berlin.   All these works are published annually several years in advance.

58. In what follows, we assume the Ephemeris to be computed for the Greenwich meridian, and, consequently, that it contains the right ascensions, declinations, equation of time, &c. for given equidistant instants of Greenwich time.

Before we can find from it the values of any of these quanti-
ties for a given local time, we must find the corresponding Green-
wich time (Arts. 45, 46). When this time is exactly one of the
instants for which the required quantity is put down in the Ephe-
meris, nothing more is necessary than to transcribe the quantity
as there put down. But when, as is mostly the case, the time
falls between two of the times in the Ephemeris, we must obtain
the required quantity by interpolation. To facilitate this inter-
polation, the Ephemeris contains the rate of change, or difference
of each of the quantities in some unit of time.

To use the difference columns with advantage, the Greenwich
time should be expressed in that unit of time for which the
difference is given: thus, when the difference is for one hour,
our time must be expressed in hours and decimal parts of an
hour; when the difference is for one minute, the time should be
expressed in minutes and decimal parts, &c.

59. *Simple interpolation.*—In the greater number of cases in
practice, it is sufficiently exact to obtain the required quantities
by *simple* interpolation; that is, by assuming that the differences
of the quantities are proportional to the differences of the times,
which is equivalent to assuming that the differences given in the
Ephemeris are constant. This, however, is never the case; but
the error arising from the assumption will be smaller the less
the interval between the times in the Ephemeris; hence, those
quantities which vary most irregularly, as the moon's right
ascension and declination, are given for every hour of Green-
wich time; others, as the moon's parallax and semidiameter, for
every twelfth hour, or for noon and midnight; others, as the
sun's right ascension, &c., for each noon; others, as the right
ascensions and declinations of the fixed stars, for every tenth day
of the year. Thus, for example, the *greatest* errors in the right
ascensions and declinations found from the American Ephe-
meris by simple interpolation are nearly as follows:—

	Error in R. A.	Error in Decl.
Sun	0ˢ.1	3″.5
Moon	0.1	1 .5
Jupiter	0.1	0 .6
Mars	0.4	2 .4
Venus	0.2	5 .4

To illustrate simple interpolation when the Greenwich time is given, we add the following

EXAMPLES.

For the Greenwich mean time 1856 March 30, 17^h 11^m 12^s, find the following quantities from the American Ephemeris: the Equation of time, the Right Ascension, Declination, Horizontal Parallax, and Semidiameter of the Sun, the Moon, and Jupiter.

1. *The Equation of time.*—The Gr. T. $=$ March 30, 17^h $11^m.2 =$ March 30, $17^h.187$.

(Page II) E at mean noon $=$	$+ 4^m 27^s.11$	H. D. $= - 0^s.763$
Corr. for $17^h.19 =$	$- 13.11$	17.19
$E = $	$+ 4\ 14.00$	$- 13.11$

NOTE.—Observe to mark E always with the sign which denotes how it is to be applied to *apparent* time. If *increasing*, the H. D. (hourly difference) should have the same sign as E; otherwise, the contrary sign.

2. *Sun's R. A. and Dec.*

(P. II.) a at $0^h =$	$0^h 36^m 40^s.78$	H. D. $+ 9^s.094$
Corr. for $17^h.187 =$	$+ 2\ 36.29$	17.187
$a = $	$0\ 39\ 17.07$	156.29
δ at $0^h =$	$+ 3° 57' 21''.9$	H. D. $+ 58''.15$
Corr. for $17^h.187 = +$	$16\ 39\ .4$	17.187
$\delta = $	$+ 4\ 14\ 1\ .3$	999.4

3. *Moon's R. A. and Dec.*

a at $17^h =$	$20^h 18^m 9^s.80$	Diff. $1^m + 2^s.4975$
Corr. for $11^m.2 =$	$+ 27\ .97$	11.2
$a = $	$20\ 18\ 37\ .77$	27.97
δ at $17^h =$	$- 25° 3' 10''.9$	Diff. $1^m + 8''.275$
Corr. for $11^m.2 =$	$+ 1\ 32\ .7$	11.2
$\delta = $	$- 25\ 1\ 38\ .2$	92.68

4. *Moon's Hor. Par. ($= \pi$) and Semid. ($= S$).*

π at $12^h =$	$58' 44''.1$	H. D. $+ 2''.17$
Corr. for $5^h.2 =$	$+ 11\ .3$	5.2
$\pi = $	$58\ 55\ .4$	11.28

$$S \text{ at } 12^h = 16' \ 2''.0 \qquad \text{Diff. in } 12^h = + \ 7''.1$$
$$\text{Corr. for } 5^h.2 = + \ 3 \ .1$$
$$S = 16 \ 5 \ .1$$

5. *Jupiter's R. A. and Dec.*

$$\alpha \text{ at } 0^h = 23^h \ 29^m \ 49^s.95 \qquad \text{H. D.} + 2^s.175$$
$$\text{Corr. for } 17^h.187 = \qquad + \ 37 \ .38 \qquad \qquad 17 \ .187$$
$$\alpha = 23 \ 30 \ 27 \ .33 \qquad \qquad 37 \ .38$$

$$\delta \text{ at } 0^h = - \ 4° \ 22' \ 45''.6 \qquad \text{H. D.} + 13''.74$$
$$\text{Corr. for } 17^h.187 = \qquad + \ 3 \ 56 \ .1 \qquad \qquad 17 \ .187$$
$$\delta = - \ 4 \ 18 \ 49 \ .5 \qquad \qquad 236.1$$

6. *Jupiter's Hor. Par. and Semid.*—At the bottom of page 231, we find for the nearest date March 31, without interpolation:

$$\pi = 1''.5 \qquad \qquad S = 15''.7$$

Note.—It may be observed that we mark hourly differences of declination *plus*, when the body is moving *northward*, and *minus* when it is moving *southward*.

In the above we have carried the computation to the utmost degree of precision ever necessary in simple interpolation.

60. *To find the right ascension and declination of the sun at the time of its transit over a given meridian, and also the equation of time at the same instant.*

When the sun is on a meridian in *west* longitude, the Greenwich *apparent* time is precisely equal to the longitude, that is, the Gr. App. T. is *after* the noon of the same date with the local date, by a number of hours equal to the longitude. When the sun is on a meridian in *east* longitude, the Gr. App. T. is *before* the noon of the same date as the local date, by a number of hours equal to the longitude. Hence, to obtain the sun's right ascension and declination and the equation of time for apparent noon at any meridian, take these quantities from the Ephemeris (page I of the month) for Greenwich Apparent Noon of the same date as the local date, and apply a correction equal to the hourly difference multiplied by the number of hours in the longitude, observing to *add* or *subtract* this correction, according as the numbers in the Ephemeris may indicate, for a time *before* or *after* noon.

EXAMPLE 1.—Longitude 167° 31′ W. 1856 March 20, App. Noon, find ⊙'s R. A., ⊙'s Dec., and Eq. of T.

Longitude = + 11ʰ 10ᵐ 4ˢ = + 11ʰ.17

a at App. 0ʰ =   0ʰ 0ᵐ 20ˢ.94   H. D. + 9ˢ.098
Corr. for + 11ʰ.17 = +   1   41 .62   +   11.17
a =   0   2   2 .56   + 101.62

δ at App. 0ʰ = +   0° 2′ 16″.5   H. D. + 59″.21
Corr. for + 11ʰ.17 = +   11   1 .4   +   11.17
δ = +   0 13 17 .9   + 661.4

E at App. 0ʰ = +   7ᵐ 31ˢ.57   H. D. − 0ˢ.759
Corr. for + 11ʰ.17 =   −   8 .48   +   11.17
E = +   7 23 .09   −   8.48

EXAMPLE 2.—Longitude 167° 31′ E. 1856 March 20, App. Noon, find ⊙'s R.A., ⊙'s Dec., and Eq. of T.

Longitude = − 11ʰ 10ᵐ 4ˢ = − 11ʰ.17

a at App. 0ʰ =   0ʰ 0ᵐ 20ˢ.94   H. D. + 9ˢ.098
Corr. for − 11ʰ.17 = −   1   41 .62   −   11.17
a = 23 58   39 .32   − 101.62

δ at App. 0ʰ = +   0° 2′ 16″.5   H. D. + 59″.21
Corr. for − 11ʰ.17 =   − 11   1 .4   −   11.17
* δ = −   0   8 44 .9   −   661.4

E at App. 0ʰ =   + 7ᵐ 31ˢ.57   H. D. − 0ˢ.759
Corr. for − 11ˢ.17 =   +   8 .48   −   11.17
E =   + 7 40 .05   +   8.48

61. *To find the mean local time of the moon's or a planet's transit over a given meridian.*

This is the same as the problem of Art. 55, in the special case where the hour angle of the moon or planet at the given meridian is 0ʰ. We can, however, obtain the required time directly from the Ephemeris, with sufficient accuracy for many purposes,

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* In this example the sun crosses the equator between the times of its transits over the local and the Greenwich meridians. The case must be noted, as it is a frequent occasion of error among navigators. The same case can occur on September 22 or 23.

by simple interpolation.   On page IV of the month (*Am. Ephem.* and *British Naut. Alm.*) we find the mean time of transit of the moon over the Greenwich meridian on each day.   This mean time is nothing more than the hour angle of the mean sun at the instant, or the difference of the right ascensions of the moon and the mean sun; and if this difference did not change, the mean *local* time of moon's transit would be the same for all meridians; but as the moon's right ascension increases more rapidly than the sun's, the moon is apparently retarded from transit to transit.   The difference between two successive times of transit given in the Ephemeris is the retardation of the moon in passing over 24^h *of longitude*, and the hourly difference given is the retardation in passing from the Greenwich meridian to the meridian 1^h from that of Greenwich.   Hence, to find the local time of the moon's transit on a given day, take the time of meridian passage from the Ephemeris for the same date (astronomical account) and apply a correction equal to the hourly difference multiplied by the longitude in hours; adding the correction when the longitude is west, subtracting it when east. The same method applies to planets whose mean times of transit are given in the Ephemeris as in the case of the moon.

EXAMPLE.—Longitude 130° 25′ E. 1856 March 22; required local time of moon's transit.

Gr. Merid. Passage March 22, $13^h.2^m.7$ H. D. $+ 1^m.59$
Corr. for Long. $- 8^h.7$ $=$ $- 13.8$ $- 8.7$
Local M. T. of transit $=$ $12\ 48.9$ $- 13.8$

62. *To find the moon's or a planet's right ascension, declination, &c., at the time of transit over a given meridian.*
Find the local time of transit by the preceding article, deduce the Greenwich time, and take out the required quantities from the Ephemeris for this time.   This is the usual nautical method, and is accurate enough even for the moon, as meridian observations of the moon at sea are not susceptible of great precision. For greater precision, find the local time by Art. 55 for $t = 0^h$, and thence the Greenwich time.   See also *Moon Culminations*, Chapter VII.

63. INTERPOLATION BY SECOND DIFFERENCES.—The differences between the successive values of the quantities given in the

$$\text{P. L. } A = \log \frac{3^h}{A} = \log 3^h - \log A$$

or, if A is in degrees,

$$\text{P. L. } A = \log \frac{3^\circ}{A} = \log 3^\circ - \log A$$

The angle is always supposed to be reduced to seconds; so that, whether A is in seconds of time or of arc, we have

$$\text{P. L. } A = \log 10800 - \log A$$

Tables of such logarithms are given in works on Navigation.

If now we wish to interpolate a value of a lunar distance for a time $T + t$ which falls between the two times of the Ephemeris T and $T + 3^h$, we are to compute the correction for the interval t and apply it to the distance given for the time T; and if we put

Δ = the difference of the distances in the Ephemeris,
Δ' = the difference in the interval t,

we shall have, by simple interpolation,

$$\Delta' = \Delta \times \frac{t}{3^h}$$

or, by logarithms,

$$\log \Delta' = \log t + \log \Delta - \log 3^h$$

or, supposing Δ, Δ', and t all reduced to seconds,

$$\log \Delta' = \log t - \text{P. L. } \Delta \qquad (62)$$

Subtracting both members of this from $\log 10800$, we have

$$\text{P. L. } \Delta' = \text{P.L. } t + \text{P. L. } \Delta \qquad (63)$$

which is computed by the tables above mentioned. By (62), however, only the common logarithmic table is required.

But the first differences of the lunar distance cannot be assumed as constant when the intervals of time are as great as 3^h. If we put

$$\text{P. L. } \Delta = Q$$

we observe that Q is variable, and the value given in the Ephemeris is to be regarded as its value at the middle instant of the interval to which it belongs. If then

Q' = the value of Q for the middle of the interval t,
ΔQ = the increase of Q in 3^h (found from the successive values in the Ephemeris),

we have

$$Q' = Q - \left(\frac{1^h.5 - \frac{1}{2}t}{3^h}\right)\Delta Q \qquad (64)$$

in which t is in hours and decimal parts. We find then, with regard to second differences,

$$\log \Delta' = \log t - Q'$$

EXAMPLE.—Find the distance d of the moon's centre from the star Fomalhaut at the Greenwich time 1856 March 30, $13^h 20^m 24^s$.

Here $T = 12^h$, $t = 1^h 20^m 24^s = 1^h.34$; $\dfrac{1^h.5 - \frac{1}{2}t}{3^h} = 0.28$; and from the Ephemeris:

March 30, 12^h (d)	$36° 17' 53''$	Q, .2993	ΔQ, $+$ .0041
Δ'	$-0\ 40\ 28$	$-$ .0011	.28
At $13^h 20^m 24^s$ $d =$	$\overline{35\ 37\ 25}$	Q', $\overline{.2982}$	$+$ .0011
		$\log t$, 3.6834	
		$\log \Delta'$, 3.3852	

66. *To find the Greenwich time corresponding to a given lunar distance on a given day.*

We find in the Ephemeris for the given day the two distances between which the given one falls; and if $\Delta' =$ difference between the first of these and the given one, $\Delta =$ difference of the distances in the Ephemeris, we find the interval t, to be added to the preceding Greenwich time, by simple interpolation, from the formula

$$t = 3^h \times \frac{\Delta'}{\Delta}$$

or

$$\log t = \log \Delta' + \text{P. L. } \Delta = \log \Delta' + Q \qquad (65)$$

and, with regard to second differences, the *true* interval, t', by the formula

$$\log t' = \log \Delta' + Q' \qquad (66)$$

where Q' has the value given in the preceding article.

But to find Q' by (64) we must first find an approximate value of t. To avoid this double computation, it is usual to find t by (65), and to give a correction to reduce it to t' in a small table which is computed as follows. We have from (64), (65), and (66)

$$\log t' - \log t = Q' - Q = -\left(\frac{1^{\text{h}}.5 - \frac{1}{2}t}{3^{\text{h}}}\right)\Delta Q$$

By the theory of logarithms, we have, M being the modulus of the common system,

$$\log x = M[(x-1) - \tfrac{1}{2}(x-1)^2 + \&c.]$$

so that

$$\log t' - \log t = \log \frac{t'}{t} = M\left[\frac{t'-t}{t} - \frac{1}{2}\left(\frac{t'-t}{t}\right)^2 + \&c.\right]$$

or, neglecting the square and higher powers of the small fraction $\dfrac{t'-t}{t}$,

$$\log t' - \log t = M\left(\frac{t'-t}{t}\right)$$

This, substituted above, gives

$$t' - t = -\frac{t(1^{\text{h}}.5 - \frac{1}{2}t)}{M \times 3^{\text{h}}}\Delta Q = -\frac{t(3^{\text{h}}-t)}{2M \times 3^{\text{h}}}\Delta Q$$

by which a table is readily computed giving the value of $t'-t$ [or the correction of t found by (65)], with the arguments ΔQ and t. In this formula t and $t'-t$ are supposed to be expressed in hours; and to obtain $t'-t$ in seconds we must multiply the second member by 3600; this will be effected if we multiply each of the factors t and $3^{\text{h}}-t$ by 60, that is, reduce them each to minutes, so that if we substitute the value of $M = .434294$ the formula becomes

$$t' - t = -\frac{t(180^{\text{m}}-t)}{2.60576}\Delta Q \qquad (67)$$

in which t is expressed in minutes, and $t'-t$ in seconds.

EXAMPLE.—1856 March 30, the distance of the moon and Fomalhaut is 35° 37′ 25″; what is the Greenwich time?

March 30, 12ʰ 0ᵐ 0ˢ $(d) = 36°$ 17′ 53″ $Q = .2993$ $\Delta Q = +41$
    $t = $ 1 20 36 $d =35$ 37 25 $\log \Delta' = 3.3852$
Ap. Gr. time $= 13$ 20 36 Δ' 40 28 $\log t = 3.6845$
By (67)*, $t'-t =$ -12
True Gr. time $= 13$ 20 24

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* Or from the "Table showing the correction required on account of the second differences of the moon's motion in finding the Greenwich time corresponding to a corrected lunar distance," which is given in the American Ephemeris, and is also included in the Tables for Correcting Lunar Distances given in Vol. II. of this work.

INTERPOLATION BY DIFFERENCES OF ANY ORDER.

67. When the exact value of any quantity is required from the Ephemeris, recourse must be had to the general interpolation formulæ which are demonstrated in analytical works. These enable us to determine intermediate values of a function from. tabulated values corresponding to equidistant values of the variable on which they depend. In the Ephemeris the data are in most cases to be regarded as functions of the time considered as the variable or argument.

Let T, $T + w$, $T + 2w$, $T + 3w$, &c., express equidistant values of the variable; F, F', F'', F''', &c., corresponding values of the given function; and let the differences of the first, second, and following orders be formed, as expressed in the following table :—

Argument.	Function.	1st Diff.	2d Diff.	3d Diff.	4th Diff.	5th Diff.	6th Diff.
T	F						
		a					
$T + w$	F'		b				
		a'		c			
$T + 2w$	F''		b'		d		
		a''		c'		e	
$T + 3w$	F'''		b''		d'		f
		a'''		c''		e'	
$T + 4w$	F^{iv}		b'''		d''		
		a^{iv}		c'''			
$T + 5w$	F^{v}		b^{iv}				
		a^{v}					
$T + 6w$	F^{vi}						

The differences are to be found by subtracting *downwards*, that is, each number is subtracted from the number below it, and the proper algebraic sign must be prefixed. The differences of any order are formed from those of the preceding order in the same manner as the first differences are formed from the given functions. The *even* differences (2d, 4th, &c.) fall in the same lines with the argument and function; the *odd* differences (1st, 3d, &c.) between the lines.

Now, denoting the value of the function corresponding to a value of the argument $T + nw$ by $F^{(n)}$, we have, from algebra,

$$F^{(n)} = F + na + \frac{n(n-1)}{1 \cdot 2} b + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} c + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} d + \&c. \quad (68)$$

in which the coefficients are those of the nth power of a binomial.

In this formula the interpolation sets out from the first of the given functions, and the differences used are the first of their respective orders. If n be taken successively equal to 0, 1, 2, 3, &c., we shall obtain the functions F, F', F'', F''', &c., and intermediate values are found by using fractional values of n. We usually apply the formula only to interpolating between the function from which we set out and the next following one, in which case n is less than unity. To find the proper value of n in each case, let $T + t$ denote the value of the argument for which we wish to interpolate a value of the function: then

$$nw = t \qquad\qquad n = \frac{t}{w}$$

that is, n is the value of t reduced to a fraction of the interval w.

EXAMPLE.—Suppose the moon's right ascension had been given in the Ephemeris for every twelfth hour as follows:

	D's R. A.	1st. Diff.	2d Diff.	3d Diff.	4th Diff.	5th Diff.
1856 March 5, 0ʰ	21ʰ 58ᵐ 28ˢ.39					
		+28ᵐ 47ˢ.04				
" 5, 12	22 27 15.43		−36ˢ.97			
		28 10.07		+4ˢ.79		
" 6, 0	22 55 25.50		32.18		+1ˢ.74	
		27 37.89		6.53		−0ˢ.66
" 6, 12	23 23 3.39		25.65		1.08	
		27 12.24		7.61		
" 7, 0	23 50 15.63		18.04			
		26 54.20				
" 7, 12	0 17 9.83					

Required the moon's right ascension for March 5, 6ʰ.

Here $T = $ March 5, 0ʰ, $t = 6^h$, $w = 12^h$, $n = \dfrac{6^h}{12^h} = \dfrac{1}{2}$; and if we denote the coefficients of a, b, c, d, e in (68) by A, B, C, D, E, we have

$$F = 21^h\ 58^m\ 28^s.39$$

$a = +\ 28^m\ 47^s.04$, $A = n$ $= \frac{1}{2}$, $Aa = +\ 14\ 23\ .52$

$b = -\ 36.97$, $B = A \cdot \dfrac{n-1}{2} = -\frac{1}{8}$, $Bb = +\ \ \ \ \ 4.62$

$c = +\ \ \ \ 4.79$, $C = B \cdot \dfrac{n-2}{3} = +\frac{1}{16}$, $Cc = +\ \ \ \ \ 0.30$

$d = +\ \ \ \ 1.74$, $D = C \cdot \dfrac{n-3}{4} = -\frac{5}{128}$, $Dd = -\ \ \ \ \ 0.07$

$e = -\ \ \ \ 0.66$, $E = D \cdot \dfrac{n-4}{5} = +\frac{7}{256}$, $Ee = -\ \ \ \ \ 0.02$

D's R. A. 1856 March 5, 6ʰ . . . . $F^{(34)} = 22\ 12\ 56\ .74$

which agrees precisely with the value given in the American Ephemeris.

68. The formula (68) may also be written as follows:

$$F^{(n)} = F + n \left(a + \frac{n-1}{2} \left(b + \frac{n-2}{3} \left(c + \frac{n-3}{4} \left(d + \frac{n-4}{5} \left(e + \&c. \right) \right) \right) \right) \right) \quad (68^*)$$

Thus, in the preceding example, we should have

$$\frac{n-4}{5} = -\tfrac{7}{10}, \qquad -\tfrac{7}{10} \times -0^s.66 \qquad = \qquad + 0^s.46$$

$$\frac{n-3}{4} = -\tfrac{5}{8}, \qquad -\tfrac{5}{8}\,(+\ 1^s.74 + 0^s.46) \quad = \qquad - 1.38$$

$$\frac{n-2}{3} = -\tfrac{1}{2}, \qquad -\tfrac{1}{2}\,(+\ 4^s.79 - 1^s.38) \quad = \qquad - 1.71$$

$$\frac{n-1}{2} = -\tfrac{1}{4}, \qquad -\tfrac{1}{4}\,(-36^s.97 - 1^s.71) \quad = \qquad + 9.67$$

$$n = \tfrac{1}{2}, \qquad \tfrac{1}{2}\,(+ 28^m 47^s.04 + 9^s.67) = + 14^m 28^s.35$$

and adding this last quantity, $14^m\ 28^s.35$, to $21^h\ 58^m\ 28^s.39$, we obtain the same value as before, or $22^h\ 12^m\ 56^s.74$.

69. A more convenient formula, for most purposes, may be deduced from (68), if we use not only values of the functions following that from which we set out, but also preceding values; that is, also values corresponding to the arguments $T - w$, $T - 2w$, &c. We then form a table according to the following schedule:

Argument.	Function.	1st Diff.	2d Diff.	3d Diff.	4th Diff.	5th Diff.	6th Diff.
$T - 3w$	$F_{\prime\prime\prime}$						
		$a_{\prime\prime\prime}$					
$T - 2w$	$F_{\prime\prime}$		$b_{\prime\prime}$				
		$a_{\prime\prime}$		$c_{\prime\prime}$			
$T - w$	$F_{\prime}$		$b_{\prime}$		$d_{\prime}$		
		$a_{\prime}$		$c_{\prime}$		$e_{\prime}$	
T	F		b		d		f
		a'		c'		e'	
$T + w$	F'		b'		d'		
		a''		c''			
$T + 2w$	F''		b''				
		a'''					
$T + 3w$	F'''						

	D 's R. A.	1st Diff.	2d Diff.	3d Diff.	4th Diff.	5th Diff.
1856 March 3, 12ʰ	20ʰ 28ᵐ 17ˢ.88					
		$+ 30ᵐ 39ˢ.20$				
" 4, 0	20 58 57.08		$− 84ˢ.27$			
		80 4.93		$− 4ˢ.28$		
" 4, 12	21 29 2.01		88.55		$+ 3ˢ.49$	
		29 26.38		$− 0.79$		$− 0ˢ.33$
" 5, 0	21 58 28.39		89.34		8.16	
		28 47.04		$+ 2.37$		$− 0.74$
" 5, 12	22 27 15.43		36.97		2.42	
		28 10.07		$+ 4.79$		
" 6, 0	22 55 25.50		82.18			
		27 87.89				
" 6, 12	23 23 8.39					

Drawing a horizontal line under the function from which we set out, the differences required in the formula (69) stand next to this line, alternately below and above it.

$$F \ = 21ʰ \ 58ᵐ \ 28ˢ.39$$

$$a' = + \ 28ᵐ \ 47ˢ.04, \quad A = \quad n \quad = \quad \tfrac{1}{2}, \quad Aa' = + \ 14 \ 23.52$$

$$b = − \quad 39.34, \quad B = A . \frac{n-1}{2} = − \ \tfrac{1}{8}, \quad Bb \ = + \quad 4.92$$

$$c' = + \quad 2.37, \quad C = B . \frac{n+1}{3} = − \ \tfrac{1}{16}, \quad Cc' \ = − \quad 0.15$$

$$d = + \quad 3.16, \quad D = C . \frac{n-2}{4} = + \ \tfrac{3}{128}, \quad Dd \ = + \quad 0.07$$

$$e' = − \quad 0.74, \quad E = D . \frac{n+2}{5} = + \ \tfrac{3}{256}, \quad Ee' \ = − \quad 0.01$$

$$\text{D's R. A. 1856 March 5, 6ʰ} \ = \ F^{(n)} = \overline{22 \ 12 \ 56.74}$$

69*. If in (69) we substitute the values

$$a' = a, + b$$
$$c' = c, + d$$
$$\&c.$$

we find

$$F^{(n)} = F + na, + \frac{(n+1)\,n}{1.2} b + \frac{(n+1)\,(n)\,(n-1)}{1.2.3} c,$$
$$+ \frac{(n+2)\,(n+1)\,(n)\,(n-1)}{1.2.3.4} d + \&c. \qquad (70)$$

in which the law of the coefficients is that one new factor is introduced into the numerator alternately *before and after* the other factors, observing still that the factors decrease by unity from left to right. The differences employed are those which lie on each side of the horizontal line drawn immediately above the function from which we set out.

	D's R. A.	1st Diff.	2d Diff.	3d Diff.	4th Diff.	5th Diff.
Mar. 3, 12ʰ	20ʰ28ᵐ17ˢ.88					
		$+30^m\ 29^s.20$				
" 4, 0	20 58 57.08		$-34^s.27$			
		30 4.93		$-4^s.28$		
" 4, 12	21 29 2.01		38.55		$+3^s.49$	
		29 26.38		-0.79		$-0^s.33$
" 5, 0	21 58 28.39	[+29 6.71]	-39.34	[+0.79]	+3.16	[−0.54]
		28 47.04		$+2.37$		-0.74
" 5, 12	22 27 15.43		36.97		2.42	
		28 10.07		$+4.79$		
" 6, 0	22 55 25.50		32.18			
		27 37.89				
" 6, 12	23 23 3.39					

Drawing two lines, one above and the other below the function from which we set out, and then filling the blanks by the means of the odd differences above and below these lines (which means are here inserted in brackets), we have presented in the same line all the differences required in the formula (71); and we then have

$$F = 21^h\ 58^m\ 28^s.39$$

$$a = +29^m\ 6^s.71,\quad A = n = \tfrac{1}{2},\quad Aa = +14\ 33.36$$

$$b = -39.34,\quad B = \frac{n^2}{2} = +\tfrac{1}{8},\quad Bb = -4.92$$

$$c = +0.79,\quad C = A\cdot\frac{n^2-1}{6} = -\tfrac{1}{16},\quad Cc = -0.05$$

$$d = +3.16,\quad D = B\cdot\frac{n^2-1}{12} = -\tfrac{1}{128},\quad Dd = -0.02$$

$$e = -0.54,\quad E = C\cdot\frac{n^2-4}{20} = +\tfrac{3}{256},\quad Ee = -0.01$$

$$F^{(x)} = 22\ 12\ 56.75$$

agreeing within 0ˢ.01 with the value found in the preceding article. Hansen has given a table for facilitating the use of this formula. (See his *Tables de la Lune*).

71. Another form, considered by Bessel as more accurate than any of the preceding, is found by employing the odd differences that fall next below the horizontal line drawn below the function from which we set out, and the means of the even differences that fall next above and next below this line. Thus, if we put

$$b_0 = \tfrac{1}{2}(b + b'),\quad d_0 = \tfrac{1}{2}(d + d'),\ \&c.$$

by $f(T + nw)$, the successive derivatives of this function corresponding to the same value of the argument will be denoted by $f'(T + nw)$, $f''(T + nw)$, $f'''(T + nw)$, &c., and $f(T)$, $f'(T)$, $f''(T)$, &c., will denote the values of the function and its derivatives corresponding to the argument T, or when $n = 0$. Hence, if we regard nw as the variable, we shall have, by Maclaurin's Theorem,

$$f(T + nw) = f(T) + f'(T)\, nw + f''(T)\frac{n^2 w^2}{1 \cdot 2} + \&c.$$

Comparing the coefficients of the several powers of n in this formula with those in (74), we have

$$f'(T) = \frac{1}{w}(a - \tfrac{1}{2}b + \tfrac{1}{3}c - \tfrac{1}{4}d + \tfrac{1}{5}e - \&c.)$$

$$f''(T) = \frac{1}{w^2}(b - c + \tfrac{11}{12}d - \tfrac{5}{6}e + \&c.)$$

$$f'''(T) = \frac{1}{w^3}(c - \tfrac{3}{2}d + \tfrac{7}{4}e - \&c.)$$

$$f^{\mathrm{iv}}(T) = \frac{1}{w^4}(d - 2e + \&c.)$$

$$f^{\mathrm{v}}(T) = \frac{1}{w^5}(e - \&c.)$$

&c.   &c.   .   .   .   .   .   .   (76)

the differences being taken as in Art. 67.

Still more convenient expressions are found by comparing Maclaurin's Theorem with (75); namely:

$$f'(T) = \frac{1}{w}(a - \tfrac{1}{6}c + \tfrac{1}{30}e - \&c.)$$

$$f''(T) = \frac{1}{w^2}(b - \tfrac{1}{12}d + \&c.)$$

$$f'''(T) = \frac{1}{w^3}(c - \tfrac{1}{4}d + \&c.)$$

$$f^{\mathrm{iv}}(T) = \frac{1}{w^4}(d - \&c.)$$

$$f^{\mathrm{v}}(T) = \frac{1}{w^5}(e - \&c.)$$

&c.   &c.   (77)

the differences being found according to the schedule in Art. 69, and the odd differences, a, c, e, &c., being interpolated means.

The preceding formulæ determine the derivatives for the value T of the argument. To find them for any other value, we have, by differentiating Maclaurin's Formula with reference to nw,

$$f'(T + nw) = f'(T) + f''(T) . nw + \tfrac{1}{2} f'''(T) . n^2 w^2 + \&c. \qquad (78)$$

in which we may substitute the values of $f'(T), f''(T)$, &c. from (76) or (77).

In like manner, by successive differentiations of (78) we obtain

$$f''(T + nw) = f''(T) + f'''(T) . nw + \tfrac{1}{2} f^{iv}(T) . n^2 w^2 + \&c.$$
$$f'''(T + nw) = f'''(T) + f^{iv}(T) . nw + \&c.$$
$$\&c. \qquad \&c.$$

76. An immediate application of (76) or (77) is the computation of *the differences in a unit of time* of the functions in the Ephemeris; for this difference is nothing more than the first derivative, denoted above by the symbol f'.

EXAMPLE.—Find the difference of the moon's right ascension in one minute for 1856 March 5, 0^h.

We have in Art. 70, for $T =$ March 5, 0^h, $a = 29^m 6^s.71$, $c = + 0^s.79$, $e = - 0^s.54$, and $w = 12^h = 720^m$. Hence, by the first equation of (77),

$$f'(T) = \tfrac{1}{720} (29^m 6^s.71 - 0^s.13 - 0^s.02) = 2^s.4258$$

On interpolation, consult also ENCKE in the Jahrbuch for 1830 and 1837.

STAR CATALOGUES.

77. The Nautical Almanac gives the position of only a small number of stars. The positions of others are to be found in the *Catalogues of stars*. These are lists of stars arranged in the order of their right ascensions, with the data from which their apparent right ascensions and declinations may be obtained for any given date.

The right ascension and declination of the so-called *fixed* stars are, in fact, ever changing: 1st, by precession, nutation, and aberration (hereafter to be specially treated of), which are not changes in the absolute position of the stars, but are either changes in the circles to which the stars are referred by spherical co-ordinates (precession and nutation), or apparent changes arising from the observer's motion (aberration); 2d, by the

$$a = \cos a \sec \delta \qquad\qquad a' = \tan \varepsilon \cos \delta - \sin a \sin \delta$$
$$b = \sin a \sec \delta \qquad\qquad b' = \cos a \sin \delta$$
$$c = 46''.078 + 20''.056 \sin a \tan \delta \quad c' = 20''.056 \cos a$$
$$d = \cos a \tan \delta \qquad\qquad d' = - \sin a$$

in which $\varepsilon =$ obliquity of the ecliptic. Or we may resort to what are usually called the *independent constants*, and dispense with the a, b, c, d, a', b', c', d' altogether, proceeding then by the formula

$$\left.\begin{aligned} a &= a_0 + \tau\mu + f + g \sin (G + a) \tan \delta + h \sin (H + a) \sec \delta \\ \delta &= \delta_0 + \tau\mu' + i \cos \delta + g \cos (G + a) + h \cos (H + a) \sin \delta \end{aligned}\right\} \quad (80)$$

the independent constants f, g, G, h, H, i being given in the Ephemeris, together with the value of τ for the given date, expressed decimally.

It should be observed that the constants a, b, c, d, a', b', c', d' are not absolutely constant, since they depend on the right ascension and declination, which are slowly changing: unless, therefore, the catalogue which contains them gives also their variations, or unless the time to which we wish to reduce is not very remote from the epoch of the catalogue, it may be preferable to use the independent constants.

In forming the products Aa, Bb, &c., attention must of course be paid to the algebraic signs of the factors. The signs of A, B, C, D are, in the Ephemerides, prefixed to their logarithms; and the signs of a, b, c, &c. are in some catalogues (as that of the British Association) also prefixed to their logarithms; but I shall here, as elsewhere in this work, mark only the logarithms of *negative* factors, prefixing to them the letter n.

It should be remarked, also, that the B. A. C.* gives the

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* B. A. C.—*British Association Catalogue*, containing 8377 stars, distributed in all parts of the heavens; a very useful work, but not of the highest degree of precision. The Greenwich Catalogues, published from time to time, are more reliable, though less comprehensive. For the places of certain fundamental stars, see Bessel's *Tabulæ Regiomontanæ* and its continuation by Wolfers and Zech.

Lalande's *Histoire Céleste* contains nearly 50,000 stars, most of which are embraced in a catalogue published by the British Association, reduced, under the direction of F. Baily, from the original work of Lalande. The Königsberg Observations embrace the series known as Bessel's Zones, the most extensive series of observations of small stars yet published. The original observations are given with data for their reduction, but an important part of them is given in Wrisse's *Positiones Mediae Stellarum fixarum in Zonis Regiomontanis a Besselio inter —15° et +15° declin. observat.*, containing nearly 32,000 stars.

See also Struve's *Catal. generalis*, and the catalogues of Argelander, Rümker,

north polar distance instead of the declination, or $\pi_0 = 90° - \delta_0$; and, since π decreases when δ increases, the corrections change their sign. This has been provided for by changing the signs of μ', a', b', c', d' in the catalogue itself, so that the computer who employs this catalogue proceeds by the formulas (79), only writing π and π_0 for δ and δ_0.

EXAMPLE.—Find the apparent right ascension and declination of a Tauri (*Aldebaran*) for June 15, 1856, from the B. A. C.

This star is B. A. C. 1420. Whence we take for

Jan. 1, 1850, Mean R. A. $= 4^h 27^m 19^s.11$ Mean N. P. D. $= 73° 47' 47''.00$

Ann. prec. $= + 3^s.428$ ⎫
Prop. motion $= + 0.008$ ⎬ for $6^{yr} = + 20.616$
$\qquad\qquad$ ⎭

$- 7''.89$ ⎫
$+ 0.15$ ⎬ $\times 6 = \qquad - 46.44$
$\qquad$ ⎭

Jan. 1, 1856, $a_0 = 4 \ 27 \ 39.726$ $\pi_0 = 73 \ 47 \ 0.56$

We next take

from B. A. C.   logs. a 8.4364 b 8.8050 c 0.5350 d 7.8821

from Ephem. ⎫
for June 15. ⎬ logs. A n0.1982 B n1.3091 C 9.5192 D n0.9027
$\qquad$ ⎭

from B. A. C.   logs. a' n9.2047 b' n9.0406 c' n0.8971 d' 9.9635

logs. Aa n8.6346 Bb n0.1141 Cc 0.0542 Dd n8.7848

logs. Aa' 9.4029 Bb' 0.3497 Cc' n0.4163 Dd' n0.8662

Corr. of a_0, $Aa = +0^s.043$, $Bb = -1^s.301$, $Cc = +1^s.133$, $Dd = -0^s.061$
Corr. of π_0, $Aa' = +0''.25$, $Bb' = +2''.29$, $Cc' = -2''.61$, $Dd' = -7''.35$

We have also from the Catalogue $\mu = + 0^s.008$, $\mu' = + 0''.15$. The fraction of the year for June 15, 1856, is $\tau = 0.45$; and hence

Jan. 1, 1856,	$a_0 = 4^h 27^m 39^s.726$		$\pi_0 =$	73° 47'	0''.56
Sum of corr. of a_0	$= -1.186$	Sum of corr. of $\pi_0 =$			-7.42
	$\tau\mu = +0.004$	$\tau\mu' =$			$+0.07$
June 15, 1856,	$a = 4 \ 27 \ 38.544$	$\pi =$		73 46 53	.21
		$\delta =$		$+16 \ 13 \ 6$	.79

78. When the greatest precision is required, we should consider the change in the star's place even in a fraction of a day, and therefore also the change while the star is passing from one meridian to another; also the secular variation and the changes

PIAZZI, SANTINI; and the published observations of the principal observatories. See also a list of catalogues in the introduction to the B. A. C.

in the precession and in the logarithms of the constants. Further, it is to be observed that the annual precession of the catalogues is for a mean year of $365^d\ 5^h.8$. But for a fuller consideration of this subject see Chapter XI.

CHAPTER III.

FIGURE AND DIMENSIONS OF THE EARTH.

79. THE apparent positions of those heavenly bodies which are within *measurable* distances from the earth are different for observers on different parts of the earth's surface, and, therefore, before we can compare observations taken in different places we must have some knowledge of the form and dimensions of the earth. I must refer the reader to geodetical works for the methods by which the exact dimensions of the earth have been obtained, and shall here assume such of the results as I shall have occasion hereafter to apply.

The figure of the earth is very nearly that of an *oblate spheroid*, that is, an ellipsoid generated by the revolution of an ellipse about its minor axis. The section made by a plane through the earth's axis is nearly an ellipse, of which the major axis is the equatorial and the minor axis the polar diameter of the earth. Accurate geodetical measurements have shown that there are small deviations from the regular ellipsoid; but it is sufficient for the purposes of astronomy to assume all the meridians to be ellipses with the mean dimensions deduced from all the measures made in various parts of the earth.

80. Let $EPQP'$, Fig. 11, be one of the elliptical meridians of the earth, EQ the diameter of the equator, PP' the polar diameter, or axis of the earth, C the centre, F a focus of the ellipse. Let

$a =$ the semi-major axis, or equatorial radius, $= CE$,
$b =$ the semi-minor axis, or polar radius, $= CP$,
$c =$ the compression of the earth,
$e =$ the eccentricity of the meridian.

By the *compression* is meant the difference of the equatorial and polar radii expressed in parts of the equatorial radius as unity, or
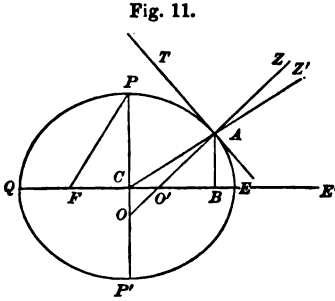
$$c = \frac{a-b}{a} = 1 - \frac{b}{a}.$$

The *eccentricity of the meridian* is the distance of either focus from the centre, also expressed in parts of the equatorial radius, or, in Fig. 11,

$$e = \frac{CF}{CE}$$

But, since $PF = CE$, we have,

$$\frac{\overline{CF}^2}{\overline{CE}^2} = \frac{\overline{PF}^2 - \overline{PC}^2}{\overline{CE}^2} = 1 - \frac{\overline{PC}^2}{\overline{CE}^2}$$

that is,

$$e^2 = 1 - \frac{b^2}{a^2} = 1 - (1-c)^2$$

or

$$e = \sqrt{2c - c^2} \qquad (81)$$

By a combination of all the most reliable measures, BESSEL deduced the most probable form of the spheroid, or that which most nearly represents all the observations that have been made in different parts of the world.   He found*

$$\frac{b}{a} = 1 - c = \frac{298.1528}{299.1528}$$

or

$$c = \frac{1}{299.1528}$$

whence, by (81),

$$e = .0816967$$

$$\log e = 8.912205 \qquad \log \sqrt{(1-ee)} = 9.9985458$$

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* *Astronomische Nachrichten*, No. 438.   See also Encke's Tables of the dimensions of the terrestrial spheroid in the *Jahrbuch* for 1852.

The absolute lengths of the semi-axes, according to BESSEL, are,

$a = 6377397.15$ metres $= 6974532.34$ yds. $= 3962.802$ miles
$b = 6356078.96$ " $= 6951218.06$ " $= 3949.555$ "

81. *To find the reduction of the latitude for the compression of the earth.*

Let A, Fig. 11, be a point on the surface of the earth; AT the tangent to the meridian at that point; AO, perpendicular to AT, the normal to the earth's surface at A. A plane touching the earth's surface at A is the plane of the horizon at that point (Art. 3), and therefore AO, which is perpendicular to that plane, represents the *vertical line* of the observer at A. This vertical line does not coincide with the radius, except at the equator and the poles. If we produce CE, OA, and CA to meet the celestial sphere in E', Z, and Z' respectively, the angle $ZO'E'$ is the declination of the zenith, or (Art. 7) the *geographical latitude*, and Z is the *geographical zenith*; the angle $Z'CE'$ is the declination of the geocentric zenith Z', and is called the *geocentric or reduced latitude;* and $ZAZ' = CAO$ is called the *reduction of the latitude.* It is evident that the geocentric is always less than the geographical latitude.

Now, if we take the axes of the ellipse as the axes of co-ordinates, the centre being the origin, and denote by x the abscissa, and by y the ordinate of any point of the curve, by a and b the semi-major and semi-minor axes respectively, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we put

$\varphi =$ the geographical latitude,
$\varphi' =$ the geocentric   "

we have, since φ is the angle which the normal makes with the axis of abscissæ,

$$\tan \varphi = -\frac{dx}{dy}$$

and from the triangle ACB,

$$\tan \varphi' = \frac{y}{x}$$

Differentiating the equation of the ellipse, we have

$$\frac{y}{x} = -\frac{b^2}{a^2} \cdot \frac{dx}{dy}$$

or

$$\tan \varphi' = \frac{b^2}{a^2} \tan \varphi = (1 - e^2) \tan \varphi \qquad (82)$$

which determines the relation between φ and φ'.

To find the difference $\varphi - \varphi'$, or the reduction of the latitude, we have recourse to the general development in series of an equation of the form

$$\tan x = p \tan y$$

which [Pl. Trig. Art. 254] is

$$x - y = q \sin 2y + \tfrac{1}{2} q^2 \sin 4y + \&c.$$

in which

$$q = \frac{p - 1}{p + 1}$$

Applying this to the development of (82), we find, after dividing by $\sin 1''$ to reduce the terms of the series to seconds,

$$\varphi - \varphi' = -\frac{q}{\sin 1''} \sin 2\varphi - \frac{q^2}{2 \sin 1''} \sin 4\varphi - \&c. \qquad (83)$$

in which

$$q = \frac{p - 1}{p + 1} = \frac{1 - e^2 - 1}{1 - e^2 + 1} = -\frac{e^2}{2 - e^2}$$

Employing BESSEL's value of e, we find

$$-\frac{q}{\sin 1''} = 690''.65 \qquad -\frac{q^2}{2 \sin 1''} = -1''.16$$

and, the subsequent terms being insensible,

$$\varphi - \varphi' = 690''.65 \sin 2\varphi - 1''.16 \sin 4\varphi \qquad (83^*)$$

by which $\varphi - \varphi'$ is readily computed for given values of φ. Its value will be found in our Table III. Vol. II. for any given value of φ.

EXAMPLE.—Find the reduced latitude when $\varphi = 35°$. We find by (83), or Table III.,

$$\varphi - \varphi' = 648''.25 = 10' \, 48''.25$$

and hence the reduced or geocentric latitude

$$\varphi' = 34° \, 49' \, 11''.75$$

82. *To find the radius of the terrestrial spheroid for a given latitude.*
Let

$$\rho = \text{the radius for the latitude } \varphi = AC.$$

We have

$$\rho = \sqrt{x^2 + y^2}$$

To express x and y in terms of φ, we have from the equation of the ellipse and its differential equation, after substituting $1 - e^2$ for $\dfrac{b^2}{a^2}$,

$$x^2 + \frac{y^2}{1 - e^2} = a^2$$

$$\frac{y}{x} = (1 - e^2) \tan \varphi$$

from which by a simple elimination we find

$$x = \frac{a \cos \varphi}{\sqrt{(1 - e^2 \sin^2 \varphi)}}$$

$$y = \frac{(1 - e^2) \, a \sin \varphi}{\sqrt{(1 - e^2 \sin^2 \varphi)}}$$

and hence

$$\rho = a \sqrt{\left[\frac{1 - 2 \, e^2 \sin^2 \varphi + e^4 \sin^2 \varphi}{1 - e^2 \sin^2 \varphi} \right]} \qquad (84)$$

by which the value of ρ may be computed. The logarithm of ρ, putting $a = 1$, is given in our Table III. Vol. II.

But the logarithm of ρ may be more conveniently found by a series. If in (84) we substitute

$$e^2 = 1 - f^2$$
$$\sin^2 \varphi = \tfrac{1}{2} (1 - \cos 2 \varphi)$$

we find, putting $a = 1$,

$$\rho = \sqrt{\left[\frac{1 + f^4 + (1 - f^4) \cos 2 \varphi}{1 + f^2 + (1 - f^2) \cos 2 \varphi} \right]}$$

$$= \frac{1 + f^2}{1 + f} \sqrt{\left[\frac{1 + \left(\dfrac{1 - f^2}{1 + f^2}\right)^2 + 2 \left(\dfrac{1 - f^2}{1 + f^2}\right) \cos 2 \varphi}{1 + \left(\dfrac{1 - f}{1 + f}\right)^2 + 2 \left(\dfrac{1 - f}{1 + f}\right) \cos 2 \varphi} \right]}$$

Now (Pl. Trig. Art. 260) if we have an expression of the form

$$X = \sqrt{(1 + m^2 - 2m \cos C)} \qquad (A)$$

Hence, also, the following:

$$\rho = a \sqrt{\left(\frac{\cos \varphi}{\cos \varphi' \cos (\varphi - \varphi')} \right)} \qquad (89)$$

83. *To find the length of the normal terminating in the axis, for a given latitude.*

Putting

$$N = \text{the normal} = AO \text{ (Fig. 11)},$$

we have evidently

$$N = \frac{\rho \cos \varphi'}{\cos \varphi} = \frac{a}{\sqrt{(1 - e^2 \sin^2 \varphi)}} \qquad (90)$$

or, employing the auxiliary ψ of the preceding article,

$$N = a \sec \psi$$

84. *To find the distance from the centre to the intersection of the normal with the axis.*

Denoting this distance by ai (so that i denotes the distance when $a = 1$), we have in Fig. 11,

$$ai = CO$$

and, from the triangle ACO,

$$ai = \frac{\rho \sin (\varphi - \varphi')}{\cos \varphi}$$

or, by (88),

$$ai = \frac{ae^2 \sin \varphi}{\sqrt{(1 - e^2 \sin^2 \varphi)}} = ae^2 \sin \varphi \sec \psi \qquad (91)$$

85. *To find the radius of curvature of the terrestrial meridian for a given latitude.*—Denoting this radius by R, we have, from the differential calculus,

$$R = \frac{[1 + (D_x y)^2]^{\frac{3}{2}}}{D_x^2 y}$$

where we employ the notation $D_x y$, $D_x^2 y$ to denote the first and second differential coefficients of y relatively to x. We have from the equation of the ellipse

$$D_x y = - \frac{b^2}{a^2} \cdot \frac{x}{y} \qquad D_x^2 y = - \frac{b^4}{a^2 y^3}$$

whence

$$R = \frac{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}}{a^4 b^4}$$

Observing that $b^2 = a^2 (1 - e^2)$, we find, by substituting the values of x and y in terms of φ (p. 99),

$$R = \frac{a (1 - e^2)}{(1 - e^2 \sin^2 \varphi)^{\frac{3}{2}}} \qquad (92)$$

EXAMPLE.—Find the radius of curvature for the latitude of Greenwich, $\varphi = 51° 28' 38''.2$, taking $a = 6377397$ metres. We find

$$R = 6373850 \text{ metres.}$$

86. *Abnormal deviations of the plumb line.*—Granting the geometrical figure of the earth to be that of an ellipsoid of revolution whose dimensions, taking the mean level of the sea, are as given in Art. 80, it must not be inferred that the direction of the plumb line at any point of the surface always coincides precisely with the normal of the ellipsoid. It would do so, indeed, if the earth were an exact ellipsoid composed of perfectly homogeneous matter, or if, originally homogeneous and plastic, it has assumed its present form solely under the influence of the attraction of gravitation combined with the rotation on its axis. But experience has shown[*] that the plumb line mostly deviates from the normal to the regular ellipsoid, not only towards the north or south, but also towards the east or west; so that the apparent zenith as indicated by the plumb line differs from the true zenith corresponding to the normal both in declination and right ascension. These deviations are due to local irregularities both in the figure and the density of the earth. Their amount is, however, very small, seldom reaching more than $3''$ of arc in any direction.

In order to eliminate the influence of these deviations at a given place, observations are made at a number of places as nearly as possible symmetrically situated around it, and, assuming the dimensions of the general ellipsoid to be as we have given them, the direction of the plumb line at the given place is deduced from its direction at each of the assumed places (by

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[*] U.S. Coast Survey Report for 1853, p. 14*.

the aid of the geodetic measures of its distance and direction from each); or, which is the same thing, the latitude and longitude of the place are deduced from those of each of the assumed places: then the mean of all the resulting latitudes is the *geodetic latitude*, and the mean of all the resulting longitudes is the *geodetic longitude*, of the place. These quantities, then, correspond as nearly as possible to the true normal of the regular ellipsoid; the geodetic latitude being the angle which this normal makes with the plane of the equator, and the geodetic longitude being the angle which the meridian plane containing this normal makes with the plane of the first meridian. The geodetic latitude is identical with the *geographical* latitude as we have defined it in Art. 81.

The *astronomical latitude* of a place is the declination of the apparent zenith indicated by the actual plumb line; but, unless when the contrary is stated, it will be hereafter understood to be identical with the geographical or geodetic latitude.

It has recently been attempted to show that the earth differs sensibly from an ellipsoid of revolution;* but no deduction of this kind can be safely made until the anomalous deviations of the plumb line above noticed have been eliminated from the discussion.

CHAPTER IV.

REDUCTION OF OBSERVATIONS TO THE CENTRE OF THE EARTH.

87. THE places of stars given in the Ephemerides are those in which the stars would be seen by an observer at the centre of the earth, and are called *geocentric*, or *true*, places. Those observed from the surface of the earth are called *observed*, or *apparent*, places.

It must be remarked, however, that the geocentric places of the Ephemeris are also called apparent places when it is intended

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* See *Astr. Nach.* No. 1308.

lines which have a common direction, *i.e.* parallel lines. But a still more direct method of comparison is obtained by referring them to one and the same straight line, as *CAZ*, *Z* being the zenith. We then call *ZCS* the true and *ZAS* the apparent zenith distance, and these are evidently the complements of the true and apparent altitudes respectively.

In the figure we have at once

$$ZAS - ZCS = ASC$$

that is, the parallax in zenith distance or altitude is the angle at the star subtended by the radius of the earth. When the star is in the horizon, as at *H'*, the radius, being at right angles to *AH'*, subtends the greatest possible angle at the star for the same distance, and this maximum angle is called the *horizontal parallax*. The *equatorial horizontal parallax* of a star is the maximum angle subtended at the star by the equatorial radius of the earth.

89. *To find the equatorial horizontal parallax of a star at a given distance from the centre of the earth.*

Let

π = the equatorial horizontal parallax,
$\varDelta$ = the given distance of the star from the earth's centre,
a = the equatorial radius of the earth,

we have from the triangle *CAH'* in Fig. 12, if *CA* is the equatorial radius,

$$\sin \pi = \frac{a}{\varDelta} \tag{93}$$

The value of π given in the Ephemeris is always that which is given by this formula when for $\varDelta$ we employ the distance of the star at the instant for which the parallax is given.

90. *To find the parallax in altitude or zenith distance, the earth being regarded as a sphere.*

Let

ζ = the true zenith distance   = *ZCS* (Fig. 12),
ζ' = the apparent zenith distance = *ZAS*,
p = the parallax in alt. or z. d.   = *CSA*.

The triangle SAC gives, observing that the angle $SAC = 180° - \zeta'$,

$$\frac{\sin p}{\sin \zeta'} = \frac{a}{\varDelta} = \sin \pi$$

or,

$$\sin p = \sin (\zeta' - \zeta) = \sin \pi \sin \zeta' \qquad (94)$$

If we put

$$h = \text{the true altitude,}$$
$$h' = \text{the apparent altitude,}$$

then it follows also that

$$\sin p = \sin (h - h') = \sin \pi \cos h' \qquad (95)$$

Except in the case of the moon, the parallax is so small that we may consider π and p to be proportional to their sines [Pl. Trig. Art. 55]; and then we have

$$p = \pi \sin \zeta' = \pi \cos h' \qquad (96)$$

Since when $\zeta' = 90°$ we have $\sin \zeta' = 1$, and when $\zeta' = 0$, $\sin \zeta' = 0$, it follows that the parallax is a maximum when the star is in the horizon, and zero when the star is in the zenith.

EXAMPLE.—Given the apparent zenith distance of Venus, $\zeta' = 64° 43'$, and the horizontal parallax $\pi = 20''.0$; find the geocentric zenith distance.

	$\log \pi$ 1.3010
$\zeta' = 64° 43' \ 0''0$	$\log \sin \zeta'$ 9.9563
$p = \qquad 18.1$	$\log p$ 1.2573
$\zeta = 64 \ \ 42 \ \ 41.9$	

When the true zenith distance is given, to compute the parallax, we may first use this true zenith distance as the apparent, and find an approximate value of p by the formula $p = \pi \sin \zeta$; then, taking the approximate value of $\zeta' = \zeta - p$, we compute a more exact value of p by the formula (94) or (96). This second approximation is unnecessary in all cases except that of the moon, and the parallax of the moon is so great that it becomes necessary to take into account the true figure of the earth, as in the following more general investigation of the subject.

91. In consequence of the spheroidal figure of the earth, the vertical line of the observer does not pass through the centre, and therefore the geocentric zenith distance cannot be directly

referred to this line. If, however, we refer it to the radius drawn from the place of observation (or CAZ', Fig. 11), the zenith distance is that measured from the geocentric zenith of the place; whereas it is desirable to use the geographical zenith. Hence we shall here consider the geocentric zenith distance to be the angle which the straight line drawn from the centre of the earth to the star makes with the straight line drawn through the centre of the earth *parallel to the vertical line of the observer*. These two vertical lines are conceived to meet the celestial sphere in the same point, namely, the geographical zenith, which is the common vanishing point of all lines perpendicular to the plane of the horizon. Thus both the true and the apparent zenith distances will be measured upon the celestial sphere from the pole of the horizon.

The azimuth of a star is, in general, the angle which a vertical plane passing through the star makes with the plane of the meridian. When such a vertical plane is drawn through the centre of the earth, it does not coincide with that drawn at the place of observation, since, by definition (Art. 3), the vertical plane passes through the vertical line, and the vertical lines are not coincident. Hence we shall have to consider a parallax in *azimuth* as well as in *zenith distance*.

92. *To find the parallax of a star in zenith distance and azimuth when the geocentric zenith distance and azimuth are given, and the earth is regarded as a spheroid.**

Let the star be referred to three co-ordinate planes at right angles to each other: the first, the plane of the horizon of the observer; the second, the plane of the meridian; the third, the plane of the prime vertical. Let the axis of x be the meridian line, or intersection of the plane of the meridian and the plane of the horizon; the axis of y, the east and west line; the axis of z, the vertical line. Let the positive axis of x be towards the south; the positive axis of y, towards the west; the positive axis of z, towards the zenith. Let

$\varDelta' =$ the distance of the star from the origin, which is the place of observation,

$\zeta' =$ the apparent zenith distance of the star,

$A' =$ the apparent azimuth   "   "   "

---

* The investigation which follows is nearly the same as that of OLBERS, to whom the method itself is due.

then, $x'\, y'$, z' denoting the co-ordinates of the star in this system, we have, by (39),

$$x' = \Delta' \sin \zeta' \cos A$$
$$y' = \Delta' \sin \zeta' \sin A$$
$$z' = \Delta' \cos \zeta'$$

Again, let the star be referred by rectangular co-ordinates to another system of planes parallel to the former, the origin now being the centre of the earth. In the celestial sphere these planes still represent the horizon, the meridian, and the prime vertical. If then in this system we put

$\Delta =$ the distance of the star from the origin,
$\zeta =$ the true zenith distance of the star,
$A =$ the true azimuth   "   "

and denote the co-ordinates of the star in this system by x, y, and z, we have, as before,

$$x = \Delta \sin \zeta \cos A$$
$$y = \Delta \sin \zeta \sin A$$
$$z = \Delta \cos \zeta$$

Now, the co-ordinates of the place of observation in this last system, being denoted by a, b, c, we have

$$a = \rho \sin (\varphi - \varphi') \qquad b = 0 \qquad c = \rho \cos (\varphi - \varphi')$$

in which $\rho =$ the earth's radius for the latitude φ of the place of observation, and φ' is the geocentric latitude, $\varphi - \varphi'$ being the reduction of the latitude, Art. 81; and the formulæ of transformation from this second system to the first are (Art. 33)

$$x = x' + a \qquad y = y' + b \qquad z = z' + c$$
or, $$x' = x - a \qquad y' = y - b \qquad z' = z - c$$

whence, substituting the above values of the co-ordinates,

$$\left.\begin{array}{l} \Delta' \sin \zeta' \cos A' = \Delta \sin \zeta \cos A - \rho \sin (\varphi - \varphi') \\ \Delta' \sin \zeta' \sin A' = \Delta \sin \zeta \sin A \\ \Delta' \cos \zeta' \quad\;\; = \Delta \cos \zeta \qquad - \rho \cos (\varphi - \varphi') \end{array}\right\} \quad (97)$$

which are the general relations between the true and apparent zenith distances and azimuths. All the quantities in the second members being given, the first two equations determine $\Delta' \sin \zeta'$, and A'; and then from this value of $\Delta' \sin \zeta'$, and that of $\Delta' \cos \zeta'$ given by the third equation, Δ' and ζ' are determined.

But it is convenient to introduce the horizontal parallax instead of $\varDelta$. For, if we put the equatorial radius of the earth $= 1$, we have

$$\sin \pi = \frac{1}{\varDelta}$$

and hence, if we divide the equations (97) by $\varDelta$, and put

$$f = \frac{\varDelta'}{\varDelta}$$

we have

$$\left.\begin{array}{l} f \sin \zeta' \cos A' = \sin \zeta \cos A - \rho \sin \pi \sin (\varphi - \varphi') \\ f \sin \zeta' \sin A' = \sin \zeta \sin A \\ f \cos \zeta' \quad\quad = \cos \zeta \quad\quad - \rho \sin \pi \cos (\varphi - \varphi') \end{array}\right\} \quad (98)$$

To obtain expressions for the difference between ζ and ζ' and between A and A', that is, for the parallax in zenith distance and azimuth, multiply the first equation of (98) by $\sin A$, the second by $\cos A$, and subtract the first product from the second; again, multiply the first by $\cos A$, the second by $\sin A$, and add the products: we find

$$\left.\begin{array}{l} f \sin \zeta' \sin (A' - A) = \rho \sin \pi \sin (\varphi - \varphi') \sin A \\ f \sin \zeta' \cos (A' - A) = \sin \zeta - \rho \sin \pi \sin (\varphi - \varphi') \cos A \end{array}\right\} \quad (99)$$

Multiplying the first of these by $\sin \frac{1}{2} (A' - A)$, the second by $\cos \frac{1}{2} (A' - A)$, and adding the products, we find, after dividing the sum by $\cos \frac{1}{2} (A' - A)$,

$$f \sin \zeta' = \sin \zeta - \rho \sin \pi \sin (\varphi - \varphi') \frac{\cos \frac{1}{2} (A' + A)}{\cos \frac{1}{2} (A' - A)}$$

which with the third equation of (98) will determine ζ'.

If we assume γ such that

$$\tan \gamma = \tan (\varphi - \varphi') \frac{\cos \frac{1}{2} (A' + A)}{\cos \frac{1}{2} (A' - A)} \quad\quad (100)$$

we have the following equations for determining ζ':

$$\left.\begin{array}{l} f \sin \zeta' = \sin \zeta - \rho \sin \pi \cos (\varphi - \varphi') \tan \gamma \\ f \cos \zeta' = \cos \zeta - \rho \sin \pi \cos (\varphi - \varphi') \end{array}\right\} \quad (101)$$

which, by the process employed in deducing (99), give

$$\left.\begin{array}{l} f \sin (\zeta' - \zeta) = \quad\quad \rho \sin \pi \cos (\varphi - \varphi') \dfrac{\sin (\zeta - \gamma)}{\cos \gamma} \\[3mm] f \cos (\zeta' - \zeta) = 1 - \rho \sin \pi \cos (\varphi - \varphi') \dfrac{\cos (\zeta - \gamma)}{\cos \gamma} \end{array}\right\} \quad (102)$$

By multiplying the first of these by $\sin \frac{1}{2} (\zeta' - \zeta)$, the second by $\cos \frac{1}{2} (\zeta' - \zeta)$, and adding the products, we find, after dividing by $\cos \frac{1}{2} (\zeta' - \zeta)$,

$$f = 1 - \frac{\rho \sin \pi \cos (\varphi - \varphi') \cos [\frac{1}{2} (\zeta' + \zeta) - \gamma]}{\cos \gamma \cos \frac{1}{2} (\zeta' - \zeta)}$$

or multiplying by $\varDelta$,

$$\varDelta' = \varDelta - \frac{\rho \cos (\varphi - \varphi') \cos [\frac{1}{2} (\zeta' + \zeta) - \gamma]}{\cos \gamma \cos \frac{1}{2} (\zeta' - \zeta)} \qquad (103)$$

The equations (99) determine rigorously the parallax in azimuth; then (100) and (102) determine the parallax in zenith distance, and (103) the distance of the star from the observer.

The relation between $\varDelta$ and $\varDelta'$ may be expressed under a more simple form. By multiplying the first of the equations (101) by $\cos \gamma$, the second by $\sin \gamma$, the difference of the products gives

$$\varDelta' = \varDelta \frac{\sin (\zeta - \gamma)}{\sin (\zeta' - \gamma)} \qquad (104)$$

93. The preceding formulæ may be developed in series. Put

$$m = \frac{\rho \sin \pi \sin (\varphi - \varphi')}{\sin \zeta}$$

then (99) become

$$f \sin \zeta' \sin (A' - A) = m \sin \zeta \sin A$$
$$f \sin \zeta' \cos (A' - A) = \sin \zeta (1 - m \cos A)$$

whence

$$\tan (A' - A) = \frac{m \sin A}{1 - m \cos A} \qquad (105)$$

and therefore [Pl. Trig. Art. 258], $A' - A$ being in seconds,

$$A' - A = \frac{m \sin A}{\sin 1''} + \frac{m^2 \sin 2A}{2 \sin 1''} + \frac{m^3 \sin 3A}{3 \sin 1''} + \&c. \qquad (106)$$

To develop γ in series, we take

$$\tan \gamma = \tan (\varphi - \varphi') \frac{\cos [A + \frac{1}{2} (A' - A)]}{\cos \frac{1}{2} (A' - A)}$$
$$= \tan (\varphi - \varphi') [\cos A - \sin A \tan \frac{1}{2} (A' - A)]$$

whence, by interchanging arcs and tangents according to the

formulæ $\tan^{-1} y = y - \frac{1}{3} y^3 + \&c.$, $\tan x = x + \frac{1}{3} x^3 + \&c.$ [Pl. Trig. Arts. 209, 213],

$$\gamma = (\varphi - \varphi') \cos A - \frac{(\varphi - \varphi')^2 \rho \sin \pi \sin^2 A \sin 1''}{2 \sin \zeta} + \&c. \quad (107)$$

where the second term of the series is multiplied by $\sin 1''$ because γ and $\varphi - \varphi'$ are supposed to be expressed in seconds.

Again, if we put

$$n = \frac{\rho \sin \pi \cos (\varphi - \varphi')}{\cos \gamma}$$

we find from (102)

$$\tan (\zeta' - \zeta) = \frac{n \sin (\zeta - \gamma)}{1 - n \cos (\zeta - \gamma)} \quad (108)$$

whence, $\zeta' - \zeta$ being in seconds,

$$\zeta' - \zeta = \frac{n \sin (\zeta - \gamma)}{\sin 1''} + \frac{n^2 \sin 2 (\zeta - \gamma)}{2 \sin 1''} + \frac{n^3 \sin 3 (\zeta - \gamma)}{3 \sin 1''} + \&c. \quad (109)$$

Adding the squares of the equations (102), we have

$$f^2 = \left(\frac{\Delta'}{\Delta}\right)^2 = 1 - 2 n \cos (\zeta - \gamma) + n^2$$

whence [equations (A) and (B), Art. 82]

$$\log \Delta' = \log \Delta - M \left(n \cos (\zeta - \gamma) + \frac{n^2 \cos 2 (\zeta - \gamma)}{2} + \&c.\right) \quad (110)$$

where $M =$ the modulus of common logarithms.

94. The second term of the series (107) is of wholly inappreciable effect; so that we may consider as exact the formula

$$\gamma = (\varphi - \varphi') \cos A \quad (111)$$

and the rigorous formulæ (105) and (108) may be readily computed under the following form:
Put

$$\sin \vartheta = m \cos A = \frac{\rho \sin \pi \sin (\varphi - \varphi') \cos A}{\sin \zeta}$$

then

$$\tan (A' - A) = \frac{\sin \vartheta \tan A}{1 - \sin \vartheta} = \tan \vartheta \tan (45° + \frac{1}{2} \vartheta) \tan A$$

$$\left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} \quad (112)$$

Put

$$\sin \vartheta' = n \cos (\zeta - \gamma) = \frac{\rho \sin \pi \cos (\varphi - \varphi') \cos (\zeta - \gamma)}{\cos \gamma}$$

then

$$\tan (\zeta' - \zeta) = \frac{\sin \vartheta' \tan (\zeta - \gamma)}{1 - \sin \vartheta'}$$
$$= \tan \vartheta' \tan (45° + \tfrac{1}{2} \vartheta') \tan (\zeta - \gamma)$$

$$(113)$$

EXAMPLE.—In latitude $\varphi = 38° 59'$ given for the moon, $A = 320° 18'$, $\zeta = 29° 30'$, and $\pi = 58' 37''.2$, to find the parallax in azimuth and zenith distance.

We have (Table III.) for $\varphi = 38° 59'$, $\varphi - \varphi' = 11' 15''$, $\log \rho = 9.999428$: hence by (111) $\gamma = 8' 39''.3$ and $\zeta - \gamma = 29° 21' 20''.7$; with which we proceed by (112) and (113) as follows:

	$\log \rho \sin \pi$	8.23118		$\log \rho \sin \pi$	8.231179
	$\log \sin (\phi - \phi')$	7.51488		$\log \cos (\phi - \phi')$	9.999998
	$\log \operatorname{cosec} \zeta$	0.30766		$\log \sec \gamma$	0.000001
	$\log \cos A$	9.88615		$\log \cos (\zeta - \gamma)$	9.940313
$\vartheta = 18''$, $\log \sin \vartheta$		5.93987	$\vartheta' = 51' 1''.5$, $\log \sin \vartheta'$		8.171491
	$\log \tan \vartheta$	5.93987		$\log \tan \vartheta'$	8.171539
	$\log \tan (45° + \tfrac{1}{2} \vartheta)$	0.00004		$\log \tan (45° + \tfrac{1}{2} \vartheta')$	0.006446
	$\log \tan A$	$n9.91919$		$\log \tan (\zeta - \gamma)$	9.750087
	$\log \tan (A' - A)$	$n5.85910$		$\log \tan (\zeta' - \zeta)$	7.928072
	$A' - A = -14''.91$			$\zeta' - \zeta = 29' 7''.79$	
	$A' = 320' 17' 45''.09$			$\zeta' = 29° 59' 7''.79$	

It is evident that we may, without a sacrifice of accuracy, omit the factors $\cos (\varphi - \varphi')$ and $\cos \gamma$ in the computation of $\sin \vartheta'$.

If we neglect the compression of the earth in this example, we find by (94) $\zeta' - \zeta = 29' 17''.8$, which is $10''$ in error.

95. *To find the parallax of a star in zenith distance and azimuth when the apparent zenith distance and azimuth are given, the earth being regarded as a spheroid.*

If we multiply the first of the equations (101) by $\cos \zeta'$ and the second by $\sin \zeta'$, the difference of the products gives

$$\sin (\zeta' - \zeta) = \frac{\rho \sin \pi \cos (\varphi - \varphi') \sin (\zeta' - \gamma)}{\cos \gamma}$$

for which, since $\cos (\varphi - \varphi')$ and $\cos \gamma$ are each nearly equal to unity, we may take, without sensible error,

$$\sin (\zeta' - \zeta) = \rho \sin \pi \sin (\zeta' - \gamma) \qquad (114)$$

in which γ has the value found by (111), or, with sufficient accuracy, by the formula

$$\gamma = (\varphi - \varphi') \cos A' \qquad (115)$$

Again, if we multiply the first of the equations (98) by $\sin A'$ and the second by $\cos A'$, the difference of the products gives

$$\sin (A' - A) = \frac{\rho \sin \pi \sin (\varphi - \varphi') \sin A'}{\sin \zeta} \qquad (116)$$

to compute which, ζ must first be found by subtracting the value of the parallax $\zeta' - \zeta$, found by (114), from the given value of ζ'.

EXAMPLE.—In latitude $\varphi = 38° 59'$, given for the moon $A' = 320° 17' 45''.09$, $\zeta' = 29° 59' 7''.79$, $\pi = 58' 37''.2$, to find the parallax in zenith distance and azimuth.

We have, as in the example Art. 94, $\varphi - \varphi' = 11' 15''$, $\log \rho = 9.999428$, $\gamma = (\varphi - \varphi') \cos A' = 8' 39''.3$, $\zeta' - \gamma = 29° 50' 28''.5$; and hence, by (114) and (116),

$\log \rho \sin \pi$	8.231179	$\log \rho \sin \pi$	8.23118
$\log \sin (\zeta' - \gamma)$	9.696879	$\log \sin (\varphi - \varphi')$	7.51488
$\log \sin (\zeta' - \zeta)$	7.928058	$\log \sin A'$	$n9.80538$
$\zeta' - \zeta =$	$29' 7''.79$	$\log \text{cosec } \zeta$	0.30766
$\zeta =$	$29° 30' 0''$	$\log \sin (A' - A)$	$n5.85910$
		$A' - A =$	$-14''.91$
		$A =$	$320° 18' 0''$

agreeing with the given values of Art. 94.

96. *For the planets or the sun,* the following formulæ are always sufficiently precise:

$$\left. \begin{array}{l} \zeta' - \zeta = \rho\pi \sin (\zeta' - \gamma) \\ A' - A = \rho\pi \sin (\varphi - \varphi') \sin A' \text{ cosec } \zeta' \end{array} \right\} (117)$$

and in most cases we may take $\zeta' - \zeta = \pi \sin \zeta'$, and $A' - A = 0$.

The quantity $\rho\pi$ is frequently called *the reduced parallax,* and $\pi - \rho\pi = (1 - \rho)\pi$ the *reduction* of the equatorial parallax for the given latitude; and a table for this reduction is given in some collections. This reduction is, indeed, sensibly the same as the correction given in our Table XIII., which will be explained more particularly hereafter. Calling the tabular correction $\Delta\pi$, we shall have, with sufficient accuracy for most purposes,

$$\rho\pi = \pi - \Delta\pi$$

97. The preceding methods of computing the parallax enable us to pass *directly* from the geocentric to the apparent azimuth and zenith distance. There is, however, an indirect method which is sometimes more convenient. This consists in reducing both the geocentric and the apparent co-ordinates to *the point in which the vertical line of the observer intersects the axis of the earth.* I shall briefly designate this point as *the point O* (Fig. 11).

We may suppose the point O to be assumed as the centre of the celestial sphere and at the same time as the centre of an imaginary terrestrial sphere described with a radius equal to the normal OA (Fig. 11). Since the point O is in the vertical line of the observer, the azimuth at this point is the same as the apparent azimuth. If, therefore, the geocentric co-ordinates are first reduced to the point O, we shall then avoid the parallax in azimuth, and the parallax in zenith distance will be found by the simple formula for the earth regarded as a sphere, taking the normal as radius.

Since the point O is in the axis of the celestial sphere, the straight line drawn from it to the star lies in the plane of the declination circle of the star; the place of the star, therefore, as seen from the point O, differs from its geocentric place only in declination, and not in right ascension. We have then only to find the reduction of the declination and of the zenith distance to the point O.

Fig. 13.
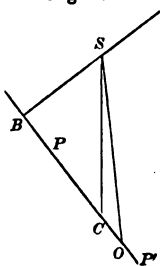
1st. *To reduce the declination to the point O.*—Let PP', Fig. 13, be the earth's axis; C the centre; O the point in which the vertical line or normal of an observer in the given latitude φ meets the axis; S the star. We have found for CO the expression (Art. 85)

$$CO = ai$$

in which a is the equatorial radius of the earth, and

$$i = \frac{e^2 \sin \varphi}{\sqrt{(1 - e^2 \sin^2 \varphi)}}$$

Let

$\varDelta$ = the star's geocentric distance   = SC,
$\varDelta_1$ = the star's distance from the point O = SO,
δ = the geocentric declination   = $90° - PCS$,
δ_1 = the declination reduced to the point O = $90° - POS$,

then, drawing SB perpendicular to the axis, the right triangles SCB and SOB give

$$\left.\begin{array}{l} \Delta_1 \sin \delta_1 = \Delta \sin \delta + ai \\ \Delta_1 \cos \delta_1 = \Delta \cos \delta \end{array}\right\} \quad (118)$$

which determine Δ_1 and δ_1. From these we deduce

$$\left.\begin{array}{l} \Delta_1 \sin (\delta_1 - \delta) = ai \cos \delta \\ \Delta_1 \cos (\delta_1 - \delta) = \Delta + ai \sin \delta \end{array}\right\} \quad (119)$$

which determine Δ_1 and the *reduction* of the declination. If we divide these by Δ, and put

$$f_1 = \frac{\Delta_1}{\Delta} \qquad \sin \pi = \frac{a}{\Delta}$$

in which π denotes, as before, the equatorial horizontal parallax, they become

$$f_1 \sin (\delta_1 - \delta) = i \sin \pi \cos \delta$$
$$f_1 \cos (\delta_1 - \delta) = 1 + i \sin \pi \sin \delta$$

whence

$$\tan (\delta_1 - \delta) = \frac{i \sin \pi \cos \delta}{1 + i \sin \pi \sin \delta}$$

or in series [Pl. Trig. Art. 257],

$$\delta_1 - \delta = \frac{i \sin \pi \cos \delta}{\sin 1''} - \frac{(i \sin \pi)^2 \sin 2\delta}{2 \sin 1''} + \&c.$$

But since the second term of the series involves i^2 and consequently e^4, and this is further multiplied by the small factor $\sin^2 \pi$, the term is wholly inappreciable even for the moon; and, as the first term cannot exceed $25''$ in any case, we shall obtain extreme accuracy by the simple formula

$$\delta_1 - \delta = i \pi \cos \delta \qquad (120)$$

The value of Δ_1 is found from (119), by the same process as was used in finding Δ' in (103), to be

$$\Delta_1 = \Delta \left\{ 1 + i \sin \pi \frac{\sin \frac{1}{2} (\delta_1 + \delta)}{\cos \frac{1}{2} (\delta_1 - \delta)} \right\}$$

or, on account of the small difference between δ_1 and δ,

$$\Delta_1 = \Delta (1 + i \sin \pi \sin \delta) \qquad (121)$$

$$AO = N = \frac{1}{\rho}$$

. If now in the vertical plane passing through the line ZO and the star S we draw SB perpendicular to OZ, and put

$$\zeta_1 = \text{the zenith distance at } O = SOZ$$
$$\zeta' = \text{the apparent zenith dist.} = SAZ$$

the triangles OSB, ASB give

$$\left. \begin{array}{l} \varDelta' \cos \zeta' = \varDelta_1 \cos \zeta_1 - \dfrac{1}{\rho} \\[2mm] \varDelta' \sin \zeta' = \varDelta_1 \sin \zeta_1 \end{array} \right\} \quad (123)$$

Dividing these equations by $\varDelta_1$, and putting

$$\frac{\varDelta'}{\varDelta_1} = f_1 \qquad \sin \pi_1 = \frac{1}{\rho \varDelta_1}$$

they become

$$f_1 \cos \zeta' = \cos \zeta_1 - \sin \pi_1$$
$$f_1 \sin \zeta' = \sin \zeta_1$$

from which we deduce

$$f_1 \sin (\zeta' - \zeta_1) = \sin \pi_1 \sin \zeta_1$$
$$f_1 \cos (\zeta' - \zeta_1) = 1 - \sin \pi_1 \cos \zeta_1$$
$$\tan (\zeta' - \zeta_1) = \frac{\sin \pi_1 \sin \zeta_1}{1 - \sin \pi_1 \cos \zeta_1} \qquad (124)$$

and in series,

$$\zeta' - \zeta_1 = \frac{\sin \pi_1 \sin \zeta_1}{\sin 1''} + \frac{\sin^2 \pi_1 \sin 2 \zeta_1}{2 \sin 1''} + \&c. \qquad (125)$$

Or, rigorously,

$$\left. \begin{array}{l} \sin \vartheta = \sin \pi_1 \cos \zeta_1 \\[2mm] \tan (\zeta' - \zeta_1) = \tan \vartheta \tan (45° + \frac{1}{2} \vartheta) \tan \zeta_1 \end{array} \right\} \quad (126)$$

To find π_1 we have

$$\sin \pi_1 = \frac{1}{\rho \varDelta_1} = \frac{1}{\rho \varDelta (1 + A \sin \pi \sin \varphi \sin \delta)}$$

or $\qquad \sin \pi_1 = \dfrac{\sin \pi}{\rho (1 + A \sin \pi \sin \varphi \sin \delta)} \qquad (127)$

But this very precise expression of π_1 will seldom be required: it will generally suffice to take

$$\sin \pi_1 = \frac{\sin \pi}{\rho}$$

or

$$\pi_1 = \frac{\pi}{\rho}$$

which will be found to give the correct value of π_1, even for the moon, within $0''.2$ in every case. Where this degree of accuracy suffices, we may employ a table containing the correction for reducing π to π_1, computed by the formula

$$\Delta \pi = \pi_1 - \pi = \pi\left(\frac{1}{\rho} - 1\right)$$

Table XIII., Vol. II., gives this correction with the arguments π and the geographical latitude φ. Taking the correction from this table, therefore, we have

$$\pi_1 = \pi + \Delta \pi \qquad (128)$$

3d. To compute the parallax in zenith distance for the point O when the apparent zenith distance is given.

Multiplying the first equation of (123) by $\sin \zeta'$, the second by $\cos \zeta'$, and subtracting, we find

$$\sin (\zeta' - \zeta_1) = \frac{1}{\rho \Delta_1} \sin \zeta'$$

or

$$\sin (\zeta' - \zeta_1) = \sin \pi_1 \sin \zeta' \qquad (129)$$

If we denote the apparent altitude by h' and the altitude reduced to the point O by h_1, this equation becomes

$$\sin (h_1 - h') = \sin \pi_1 \cos h' \qquad (130)$$

EXAMPLE.—In Latitude $\varphi = 38° 59'$, given the moon's hour angle $t = 341° 1' 36''.85$, geocentric declination $\delta = + 14° 39' 24''.54$, and the equatorial horizontal parallax $\pi = 58' 37''.2$, to find the apparent zenith distance and azimuth.

The geocentric zenith distance and azimuth, computed from these data by Art. 14, are $\zeta = 29° 30'$, $A = 320° 18'$, which are the values employed in our example in Art. 94. To compute

by the method of the present article, we first reduce the declination to the point O by (122), as follows:

For $\varphi = 38° 59'$	$\log A$	7.8250
$\pi = 3517''.2$	$\log \pi$	3.5462
	$\log \sin \varphi$	9.7987
$\delta = 14° 39' 24''.54$	$\log \cos \delta$	9.9856
$\delta_1 - \delta =$ 14 .31	$\log (\delta_1 - \delta)$ 1.1555	
$\delta_1 = 14° 39' 38''.85$		

With this value of δ_1 and $t = 341° 1' 36''.85$, the computation of the zenith distance and azimuth by Art. 14 gives for the point O

$$\zeta_1 = 29° 29' 47''.67 \qquad A_1 = 320° 17' 45''.09$$

and this value of A_1 is precisely the same as A' found in Art. 94, as it should be, since the azimuth at the point O and at the observer are identical.

We find from Table XIII. $\Delta\pi = 4''.6$, and hence $\pi = 58' 37''.2 + 4''.6 = 58' 41''.8$; and then, by (126),

	$\log \sin \pi_1$	8.23232
	$\log \cos \zeta_1$	9.93971
$\vartheta = 51' 5''$	$\log \sin \vartheta$	8.17203
	$\log \tan \vartheta$	8.17208
	$\log \tan (45° + \frac{1}{2}\vartheta)$	0.00645
$\zeta_1 = 29° 29' 47''.67$	$\log \tan \zeta_1$	9.75258
$\zeta' - \zeta_1 =$ 29 20 .03	$\log \tan (\zeta' - \zeta_1)$	7.93111
$\zeta' = 29° 59' 7''.70$		

agreeing with the value found in Art. 94 within $0''.09$. If we had computed π_1 by (127), the agreement would have been exact.

98. *To find the parallax of a star in right ascension and declination when its geocentric right ascension and declination are given.*

The investigation of this problem is similar to that of Art. 92. Let the star be referred by rectangular co-ordinates to three planes passing through the centre of the earth: the first, the plane of the equator; the second, that of the equinoctial colure; the third, that of the solstitial colure. Let the axis of x be the straight line drawn through the equinoctial points, positive towards the vernal equinox; the axis of y, the intersection of

the plane of the solstitial colure and that of the equator, positive towards that point of the equator whose right ascension is 90°; the axis of z, the axis of the heavens, positive towards the north. Let

$a =$ the star's geocentric right ascension,
$\delta =$ "   "   declination,
$\Delta =$ "   "   distance,

then the co-ordinates of the star are

$$x = \Delta \cos \delta \cos a$$
$$y = \Delta \cos \delta \sin a$$
$$z = \Delta \sin \delta$$

Again, let the star be referred to another system of planes parallel to the first, the origin being the observer. The vanishing circles of these planes in the celestial sphere are still the equator, the equinoctial colure, and the solstitial colure. Let

$a' =$ the star's observed right ascension,
$\delta' =$ "   "   declination,
$\Delta' =$ "   distance from the observer,

where by *observed* right ascension and declination we now mean the values which differ from the geocentric values by the parallax depending on the position of the observer on the surface of the earth. The co-ordinates of the star in this system will be

$$x' = \Delta' \cos \delta' \cos a'$$
$$y' = \Delta' \cos \delta' \sin a'$$
$$z' = \Delta' \sin \delta'$$

Now, if

$\Theta =$ the sidereal time $=$ the right ascension of the observer's meridian at the instant of observation,
$\varphi' =$ the reduced latitude of the place of observation,
$\rho =$ the radius of the earth for this latitude,

then Θ, φ', and ρ are the polar co-ordinates of the observer, entirely analogous to α, δ, and Δ of the star, so that the rectangular co-ordinates of the observer, taken in the first system, are

$$a = \rho \cos \varphi' \cos \Theta$$
$$b = \rho \cos \varphi' \sin \Theta$$
$$c = \rho \sin \varphi'$$

ascension or $\alpha' - \alpha$; (136) the parallax in declination, or $\delta' - \delta$; and (137) determines Δ'.

99. To obtain the developments in series, put

$$m = \frac{\rho \sin \pi \cos \varphi'}{\cos \delta}$$

then from (133) we have

$$\tan (\alpha' - \alpha) = \frac{m \sin (\alpha - \Theta)}{1 - m \cos (\alpha - \Theta)} \qquad (138)$$

whence

$$\alpha' - \alpha = \frac{m \sin (\alpha - \Theta)}{\sin 1''} + \frac{m^2 \sin 2 (\alpha - \Theta)}{2 \sin 1''} + \&c. \qquad (139)$$

Putting

$$n = \frac{\rho \sin \pi \sin \varphi'}{\sin \gamma}$$

we have from (136)

$$\tan (\delta' - \delta) = \frac{n \sin (\delta - \gamma)}{1 - n \cos (\delta - \gamma)} \qquad (140)$$

whence

$$\delta' - \delta = \frac{n \sin (\delta - \gamma)}{\sin 1''} + \frac{n^2 \sin 2 (\delta - \gamma)}{2 \sin 1''} + \&c. \qquad (141)$$

100. The quantity $\alpha - \Theta$ is the hour angle of the star east of the meridian. According to the usual practice, we shall reckon the hour angle towards the west, and denote it by t, or put

$$t = \Theta - \alpha$$

and then we shall write (138) and (140) as follows:

$$\tan (\alpha - \alpha') = \frac{m \sin t}{1 - m \cos t}$$

$$\tan (\delta - \delta') = \frac{n \sin (\gamma - \delta)}{1 - n \cos (\gamma - \delta)}$$

The rigorous computation will be conveniently performed by the following formulæ:

$$\sin \vartheta = m \cos t = \frac{\rho \sin \pi \cos \varphi' \cos t}{\cos \delta}$$

$$\tan (\alpha - \alpha') = \tan \vartheta \tan (45° + \tfrac{1}{2} \vartheta) \tan t$$

$$\tan \gamma = \frac{\tan \varphi' \cos \tfrac{1}{2} (\alpha - \alpha')}{\cos [t + \tfrac{1}{2} (\alpha - \alpha')]}$$ (142)

$$\sin \vartheta' = n \cos (\gamma - \delta) = \frac{\rho \sin \pi \sin \varphi' \cos (\gamma - \delta)}{\sin \gamma}$$

$$\tan (\delta - \delta') = \tan \vartheta' \tan (45° + \tfrac{1}{2} \vartheta') \tan (\gamma - \delta)$$

101. Except for the moon, the first terms of the series (139) and (141) will suffice, and we may use the following approximations:

$$\alpha - \alpha' = \frac{\rho \pi \cos \varphi' \sin t}{\cos \delta}$$

$$\tan \gamma = \frac{\tan \varphi'}{\cos t}$$ (143)

$$\delta - \delta' = \frac{\rho \pi \sin \varphi' \sin (\gamma - \delta)}{\sin \gamma}$$

If the star is on the meridian, we have $t = 0$, and hence $\gamma = \varphi'$, and

$$\delta - \delta' = \rho \pi \sin (\varphi' - \delta)$$

Since in the meridian we have $\zeta = \varphi - \delta$, it is easily seen that $\zeta' - \zeta$ found by (108) and $\delta' - \delta$ found by (140) will then be numerically equal, or *the parallax in zenith distance is numerically equal to the parallax in declination when the star is on the meridian.*

102. *To find the parallax of a star in right ascension and declination, when its observed right ascension and declination are given.*

Multiplying the first equation of (132) by $\sin \alpha'$, the second by $\cos \alpha'$, and subtracting one product from the other, we find

$$\sin (\alpha - \alpha') = \frac{\rho \sin \pi \cos \varphi' \sin (\Theta - \alpha')}{\cos \delta}$$

In like manner, from (135) we deduce

$$\sin(\delta - \delta') = \frac{\rho \sin \pi \sin \varphi' \sin(\gamma - \delta')}{\sin \gamma}$$

We have here $\Theta - \alpha'$ equal to the apparent or observed hour angle; and hence, putting

$$t' = \Theta - \alpha'$$

the computation may be made under the following form:

$$\left.\begin{array}{c} \sin(\alpha - \alpha') = \dfrac{\rho \sin \pi \cos \varphi' \sin t'}{\cos \delta} \\[2ex] \tan \gamma = \dfrac{\tan \varphi' \cos \frac{1}{2}(\alpha - \alpha')}{\cos[t' - \frac{1}{2}(\alpha - \alpha')]} \\[2ex] \sin(\delta - \delta') = \dfrac{\rho \sin \pi \sin \varphi' \sin(\gamma - \delta')}{\sin \gamma} \end{array}\right\} \quad (144)$$

In the first computation of $\alpha - \alpha'$ we employ δ' for δ. The value of $\alpha - \alpha'$ thus found is sufficiently exact for the computation of γ and $\delta - \delta'$. With the computed value of $\delta - \delta'$ we then find δ and correct the computation of $\alpha - \alpha'$.

EXAMPLE.—Suppose that on a certain day at the Greenwich Observatory the right ascension and declination of the moon were observed to be

$$\alpha' = 7^h 41^m 20^s.436$$
$$\delta' = 15° 50' 27''.66$$

when the sidereal time was

$$\Theta = 11^h 17^m 0^s.02$$

and the moon's equatorial horizontal parallax was

$$\pi = 56' 57''.5$$

Required the geocentric right ascension and declination.

We have for Greenwich $\varphi = 51° 28' 38''.2$, and hence (Table III.) $\varphi - \varphi' = 11' 13''.6$, $\varphi' = 51° 17' 24''.6$, $\log \rho = 9.9991134$. The computation by (144) is then as follows:

a' (in arc) $= 115°\ 20'\ \ 6''.54$ log $\rho \sin \pi$ 8.218377

Θ " $= 169\ \ 15\ \ \ 0\ .30$ log cos ϕ' 9.796142

        $t' = \overline{\ \ 53\ \ 54\ \ 53\ .76}$ log sin t' 9.907489

$\frac{1}{2}\,(a - a') = $ $14\ \ 55\ .8$ (1)   7.922008

$t' - \frac{1}{2}\,(a - a') = $ $53\ \ 39\ \ 58$ log cos δ' 9.983185

log sec $[t' - \frac{1}{2}(a - a')]$ 0.227319   App. log sin $(a - a')$ 7.938823

log cos $\frac{1}{2}\,(a - a')$ 9.999996   Approx. $a - a' = $ $29'\ 51''.6$

log tan ϕ' 0.096133   (1)   .   .   7.922008

log tan γ 0.323448   log cos δ 9.981835

            $\gamma = $ $64°\ 35'\ 57''.6$ log sin $(a - a')$ 7.940173

       $\gamma - \delta' = $ $48\ \ 45\ \ 30$ $a - a' = +\ 29'\ 57''.23$

log $\rho \sin \pi$ 8.218377 $a = 115°\ 50'\ \ 8''.77$

log sin ϕ' 9.892275 $= 7^h\ 43^m\ 20^s.251$

log sin $(\gamma - \delta')$ 9.876181

log cosec γ 0.044153

log sin $(\delta - \delta')$ 8.030986

      $\delta - \delta' = +\ 36'\ 55''.24$

         $\delta = 16°\ 27'\ 22''.90$

103. For all bodies except the moon, the second computation will never affect the result in a sensible degree, and we may use the following approximations:

$$
\left.
\begin{aligned}
a - a' &= \frac{\rho\pi \cos \varphi' \sin t'}{\cos \delta'} \\[2mm]
\tan \gamma &= \frac{\tan \varphi'}{\cos t'} \\[2mm]
\delta - \delta' &= \frac{\rho\pi \sin \varphi' \sin (\gamma - \delta')}{\sin \gamma}
\end{aligned}
\right\} \quad (145)
$$

For the sun, planets, and comets, it is frequently more convenient to use the geocentric distance of the body instead of the parallax, or, at least, to deduce the parallax from the distance, the latter being given. This distance is always expressed in parts of the sun's mean distance as unity. If we put

    $\pi_0 = $ the sun's mean equatorial horizontal parallax,

    $\varDelta_0 = $ the sun's mean distance from the earth,

we have, whatever unit is employed in expressing $\varDelta$, $\varDelta_0$, and a,

$$
\sin \pi = \frac{a}{\varDelta} \qquad\qquad \sin \pi_0 = \frac{a}{\varDelta_0}
$$

whence

$$\sin \pi = \frac{\varDelta_0}{\varDelta} \sin \pi_0$$

and when we take $\varDelta_0 = 1$,

$$\sin \pi = \frac{\sin \pi_0}{\varDelta} \qquad \text{or } \pi = \frac{\pi_0}{\varDelta} \qquad\qquad (146)$$

According to ENCKE's determination

$$\pi_0 = 8''.57116 \qquad \log \pi_0 = 0.93304$$

EXAMPLE.—DONATI's comet was observed by Mr. JAMES FER-
GUSON at Washington, 1858 Oct. 13, 6^h 26^m $21^s.1$ mean time,
and its observed right ascension and declination when corrected
for refraction were

$$\alpha' = 236° \; 48' \;\; 0''.5$$
$$\delta' = -7° \; 36' \; 52''.8$$

The logarithm of the comet's distance from the earth was $\log \varDelta$
$= 9.7444$. Required the geocentric place.

We have for Washington $\varphi = 38° \; 53' \; 39''.3$, whence, by Table
III., $\log \rho \cos \varphi' = 9.8917$, $\log \rho \sin \varphi' = 9.7955$. Converting the
mean into sidereal time (Art. 50), we find $\Theta = 19^h \; 55^m \; 16^s.98$.
Hence, by (145) and (146),

$\Theta =$	298° 49'.2	$\log \tan \varphi'$	9.9038
$\alpha' =$	236 48 .0	$\log \cos t'$	9.6713
$t' =$	62 1 .2	$\log \tan \gamma$	0.2325
$\log \pi_0$	0.9330	γ	$= 59° \; 39'.2$
$\log \varDelta$	9.7444	$\gamma - \delta'$	$= 67 \;\; 16 .1$
$\log \pi$	1.1886		
$\log \rho \pi \cos \varphi'$	1.0803	$\log \rho \pi \sin \varphi'$	0.9841
$\log \sin t'$	9.9460	$\log \sin (\gamma - \delta')$	9.9649
$\log \sec \delta'$	0.0038	$\log \operatorname{cosec} \gamma$	0.0640
$\log (\alpha - \alpha')$	1.0301	$\log (\delta - \delta')$	0.0130
$\alpha - \alpha' = + 10''.7$		$\delta - \delta' = + 10''.3$	

Hence, for the geocentric place of the comet,

$$\alpha = 236° \; 48' \; 11''.2 \qquad \delta = -7° \; 36' \; 42''.5$$

104. *Parallax in latitude and longitude.*—Formulæ similar to the
above obtain for the parallax in latitude and longitude. We

direction of a ray of light, the path of the ray will be at first a straight line; but upon entering the atmosphere its direction will be changed. According to the second law above stated, the new medium being the denser, the ray will be bent towards the normal, which in this case is a line drawn from the centre of the earth to the surface of the atmosphere at the point of incidence.

The atmosphere, however, is not of uniform density, but is most dense near the surface of the earth, and gradually decreases in density to its upper limit, where it is supposed to be of such extreme tenuity that its first effect upon a ray of light may be considered as infinitesimal. The ray is therefore *continually* passing from a rarer into a denser medium, and hence its direction is continually changed, so that its path becomes a curve which is concave towards the earth.

The last direction of the ray, or that which it has when it reaches the eye, is that of a tangent to its curved path at this point; and the difference of the direction of the ray before entering the atmosphere and this last direction is called the *astronomical refraction*, or simply the refraction.

Thus, Fig. 16, the ray *Se* from a star, entering the atmosphere at *e*, is bent into the curve *ecA* which reaches the observer at *A* in the direction of the tangent *S'A* drawn to the curve at *A*. If *CAZ* is the vertical line of the observer, or normal at *A*, by the first law of the preceding article, the vertical plane of the observer which contains the tangent *AS'* must also contain the whole curve *Ae* and the incident ray *Se*. Hence refraction increases the apparent altitude of a star, but does not affect its azimuth.

Fig. 16.

The angle *S'AZ* is the *apparent zenith distance* of the star. The *true* zenith distance* is strictly the angle which a straight line drawn from the star to the point *A*

---
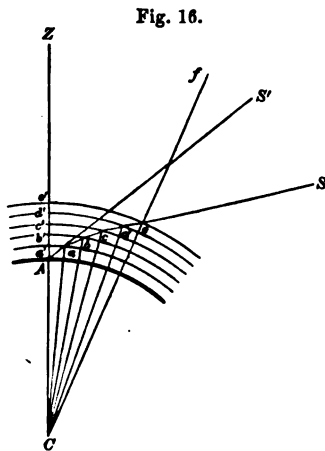
* By *true* zenith distance we here (and so long as we are considering only the effect of refraction) mean that which differs from the apparent zenith distance only by the refraction.

TABLE II. is BESSEL's Refraction Table,[*] which is generally regarded as the most reliable of all the tables heretofore constructed. In Column A of this table the refraction is regarded as a function of the *apparent* zenith distance z, and the adopted form of this function is

$$r = \alpha \beta^A \gamma^\lambda \tan z$$

in which α varies slowly with the zenith distance, and its logarithm is therefore readily taken from the table with the argument z. The exponents A and λ differ sensibly from unity only for great zenith distances, and also vary slowly; their values are therefore readily found from the table.

The factor β depends upon the barometer. The actual pressure indicated by the barometer depends not only upon the height of the column, but also upon its temperature. It is, therefore, put under the form

$$\beta = BT$$

and log B and log T are given in the supplementary tables with the arguments "height of the barometer," and "height of the attached thermometer," respectively; so that we have

$$\log \beta = \log B + \log T$$

Finally, log γ is given directly in the supplementary table with the argument "external thermometer." This thermometer must be so exposed as to indicate truly the temperature of the atmosphere at the place of observation.

In Column B of the table the refraction is regarded as a function of the *true* zenith distance ζ expressed under the form

$$r = \alpha' \beta^{A'} \gamma^{\lambda'} \tan \zeta$$

and log α', A', and λ' are given in the table with the argument ζ; β and γ being found as before.

Column A will be used when z is given to find ζ; and Column B, when ζ is given to find z.

Column C is intended for the computation of differential refraction, or the difference of refraction corresponding to small

---

[*] From his *Astronomische Untersuchungen*, Vol. I.

$$\text{log } B = + 0.00253 \qquad \text{log } \gamma = + 0.04545$$
$$\text{log } T = + 0.00127$$
$$\text{log } \beta = + 0.00380$$

Hence the refraction is computed as follows:

$$
\begin{aligned}
\text{log } a &= 1.74981 \\
A \text{ log } \beta = \text{log } \beta^A &= + 0.00381 \\
\lambda \text{ log } \gamma = \text{log } \gamma^\lambda &= + 0.04694 \\
\text{log tan } z &= 0.69154 \\
r = 310''.53 = 5' \, 10''.53 \quad \text{log } r &= 2.49210
\end{aligned}
$$

The true zenith distance is, therefore, 78° 30′ 0″ + 5′ 10″.53 = 78° 35′ 10″.53.

EXAMPLE 2.—Given the true zenith distance $\zeta = 78°$ 35′ 10″.53, Barom. 29.770 inches, Attached Therm. — 0°.4 F., External Therm. — 2°.0 F.

We find from Table II., Col. B, for 78° 35′ 10″,

$$\text{log } a' = 1.74680 \qquad A' = 0.9967 \qquad \lambda' = 1.0261$$

and from the tables for barometer and thermometer, as before,

$$\text{log } B = + 0.00253 \qquad \text{log } \gamma = + 0.04545$$
$$\text{log } T = + 0.00127$$
$$\text{log } \beta = + 0.00380$$

The refraction is then computed as follows:

$$
\begin{aligned}
\text{log } a' &= 1.74680 \\
A' \text{ log } \beta = \text{log } \beta^{A'} &= + 0.00379 \\
\lambda' \text{ log } \gamma = \text{log } \gamma^{\lambda'} &= + 0.04663 \\
\text{log tan } \zeta &= 0.69489 \\
r = 310''.53'' = 5' \, 10''.53 \quad \text{log } r &= 2.49211
\end{aligned}
$$

and the apparent zenith distance is therefore 78° 30′.

EXAMPLE 3.—Given $z = 87°$ 30′, barometer and thermometer as in the preceding examples.

By the supplementary table above given,

$$
\begin{aligned}
&&& \text{log } R = 2.98269 \\
A = 1.0298 \quad &\text{log } \beta = + 0.00380 \quad & \text{log } \beta^A &= + 0.00381 \\
\lambda = 1.2624 \quad &\text{log } \gamma = + 0.04545 \quad & \text{log } \gamma^\lambda &= + 0.05738 \\
&r = 18' \, 26''.3 \quad & \text{log } r &= 3.04388
\end{aligned}
$$

fracted in the directions *ed*, *dc*, &c. to the point *A*. The last direction of the ray is *aA*, which, when the number of strata is supposed to be infinite, becomes a tangent to the curve *ecA* at *A*, and consequently *AaS'* is the apparent direction of the star. Let the normals *Ce*, *Cd*, &c. be drawn to the successive strata. The angle *Sef* is the first angle of incidence, the angle *Ccd* the first angle of refraction. At any intermediate point between *e* and *A*, as *c*, we have *Ccd*, the supplement of the angle of incidence, and *Ccb*, the angle of refraction.

If now for any point, as *c*, in the path of the ray, we put

i = the angle of incidence,
f = the angle of refraction,
μ = the index of refraction for the stratum above *c*,
μ' = " " " below *c*,

then, Art. 105,

$$\frac{\sin i}{\sin f} = \frac{\mu'}{\mu} \qquad (148)$$

If we put

q = the normal *Cc* to the upper of the two strata,
q' = " *Cb* " lower " "
i' = the angle of incidence in the lower stratum,
  = 180° − *Cbc*,

the rectilinear triangle *Cbc* gives

$$\frac{\sin i'}{\sin f} = \frac{q}{q'}$$

which, with the above proportion, gives

$$q \mu \sin i = q'\mu' \sin i'$$

an equation which shows that the product of the normal to any stratum by its index of refraction and the sine of the angle of incidence is the same for any two consecutive strata; that is, it is a constant product for all the strata. If then we put

z = the apparent zenith distance,
a = the normal at the observer, or radius of the earth,
μ_0 = the index of refraction of the air at the observer,

we have, since *z* is the angle of incidence at the observer,

$$q\mu \sin i = a\mu_0 \sin z \qquad (149)$$

$$\left(\frac{\mu}{\mu_0}\right)^{n+1} = \frac{a}{q} \tag{151}$$

which with (149) gives

$$\sin i = \left(\frac{\mu}{\mu_0}\right)^n \sin z \tag{152}$$

or, logarithmically,

$$\log \sin i = n \log \mu + \log\left(\frac{\sin z}{\mu_0^n}\right)$$

where the last term is constant. By differentiation, therefore,

$$\frac{di}{\tan i} = n \frac{d\mu}{\mu}$$

which with (150) gives

$$dr = \frac{di}{n}$$

and, integrating,

$$r = \frac{i}{n} + C$$

To determine the constant C, the integral is to be taken from the upper limit of the atmosphere to the surface of the earth. At the upper limit $r = 0$; and if we put $\vartheta =$ the value of i at that limit, we have

$$0 = \frac{\vartheta}{n} + C$$

At the lower limit the value of r is the whole atmospheric refraction, and $i = z$: hence

$$r = \frac{z}{n} + C$$

Eliminating the constant, we have

$$r = \frac{z - \vartheta}{n} \tag{153}$$

To find ϑ, we have, by putting $\mu = 1$ in (152), since the density of the air at the upper limit is to be taken as zero,

$$\sin \vartheta = \frac{\sin z}{\mu_0^n} \tag{154}$$

Having then found μ_0 at the surface of the earth and suitably

is known from experiment; so that the true relation between the pressures and densities at different temperatures is expressed by the known formula

$$\frac{p}{p_0} = \frac{\delta}{\delta_0} [1 + \epsilon (\tau - \tau_0)]$$

whence

$$\frac{p\delta_0}{p_0\delta} = 1 + \epsilon (\tau - \tau_0) \qquad (171)$$

which combined with (169) gives

$$x = 2l\epsilon (\tau_0 - \tau)$$

and hence equal increments of x correspond to equal decrements of τ. Hence, *in this hypothesis, the heat of the strata of the atmosphere decreases as their density in arithmetical progression.* The value of ϵ, according to RUDBERG and REGNAULT, is very nearly $\frac{1}{273}$. Hence we must ascend to a height $\frac{2l}{273} = 58.6$ metres, in order to experience a decrease of temperature of 1° C. But, according to the observations of GAY LUSSAC in his celebrated balloon ascension at Paris (in the year 1804), the decrease of temperature was 40°.25 C. for a height of 6980 metres, or 1° C. for 173 metres, so that in the hypothesis under consideration the height is altogether too small, or the decrease of temperature is too rapid. This hypothesis, therefore, is not sustained either by the observed refraction or by the observed law of the decrease of temperature.

112. *Second hypothesis.*—Before proposing a new hypothesis, let us determine the relation between the height and the density of the air at that height, when the atmosphere is assumed to be throughout of the same temperature, in which case we should have the condition (170). Resuming the differential equation (161),

$$dp = g_0 a\delta \, d\left(\frac{a}{a + x}\right)$$

put

$$\frac{a}{a + x} = 1 - s$$

in which s is a new variable very nearly proportional to x. We then have

$$dp = - g_0 a\delta ds$$

which with the supposition (170) gives

$$\frac{dp}{p} = - \frac{g_0 \delta_0 a\, ds}{p_0}$$

Integrating,

$$\log p = - \frac{g_0 \delta_0}{p_0} as + C$$

in which the logarithm is Napierian. The constant being determined so that p becomes p_0 when $s = 0$, we have

$$\log p_0 = C$$

and therefore

$$\log \frac{p}{p_0} = - \frac{g_0 \delta_0}{p_0} as = - \frac{as}{l}$$

where l has the value (163). Hence, e being the Napierian base,

$$\frac{p}{p_0} = \frac{\delta}{\delta_0} = e^{-\frac{as}{l}} \tag{172}$$

which is the expression of the law of decreasing densities upon the supposition of a uniform temperature. In our first hypothesis the temperatures decrease, but nevertheless too rapidly. *We must, then, frame an hypothesis between that and the hypothesis of a uniform temperature.*

Now, in our first hypothesis we have by (169), within terms involving the second and higher powers of s,

$$\frac{p\delta_0}{p_0\delta} = 1 - \frac{as}{2l}$$

and in the hypothesis of a uniform temperature,

$$\frac{p\delta_0}{p_0\delta} = 1$$

The arithmetical mean between these would be

$$\frac{p\delta_0}{p_0\delta} = 1 - \frac{as}{l}$$

but, as we have no reason for assuming exactly the arithmetical mean, BESSEL proposes to take

$$\frac{p\delta_0}{p_0\delta} = e^{-\frac{as}{h}} = 1 - \frac{as}{h} + \frac{1}{2}\left(\frac{as}{h}\right)^2 - \&c. \tag{173}$$

h being a new constant *to be determined so as to satisfy the observed refractions.* This equation, which we shall adopt as our second hypothesis, expresses the assumed law of decreasing temperatures, since, by (171), it amounts to assuming

$$1 + \epsilon\,(\tau - \tau_0) = e^{-\frac{as}{h}} \tag{174}$$

and it follows that in this hypothesis the temperatures will not decrease in arithmetical progression with increasing heights, though they will do so very nearly for the smaller values of s, that is, near the earth's surface.

Now, combining the supposition (173) with the equation

$$dp = -\,g_0 a\delta\,ds$$

we have

$$\frac{dp}{p} = -\,\frac{g_0\delta_0 a}{p_0}\,e^{-\frac{as}{h}}\,ds = -\,\frac{a}{l}\,e^{\frac{as}{h}}\,ds$$

Integrating and determining the constant so that for $s = 0$, p becomes p_0, we have

$$\frac{p}{p_0} = e^{-\frac{h}{l}\,(e^{\frac{as}{h}}-1)}$$

which with (173) gives*

$$\delta = \delta_0\,e^{-\frac{h}{l}\,(e^{\frac{as}{h}}-1)+\frac{as}{h}}$$

Inasmuch as the law of the densities thus expressed is still hypothetical, we may simplify the exponent of e. For if h is much greater than l (as is afterwards shown), we may in this exponent put $e^{\frac{as}{h}} - 1 = \frac{as}{h}$ and we shall have as the expression of our hypothesis

$$\delta = \delta_0\,e^{-\frac{as}{l}+\frac{as}{h}} = \delta_0\,e^{-\frac{h-l}{h}\cdot\frac{as}{l}} \tag{175}$$

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* BESSEL. Fundamenta Astronomiæ, p. 28.

By comparing this with (172), we see that this new hypothesis differs from that of a uniform temperature by the correction $\frac{as}{h}$ applied to the exponent of e.

Putting, for brevity,

$$\beta = \frac{h-l}{h} \cdot \frac{a}{l} \qquad (176)$$

we have

$$\delta = \delta_0 \, e^{-\beta s} \qquad (177)$$

in which β is constant. This expression of the density is to be introduced into the differential equation of the refraction (150).

Now, by (149), in which $q = a + x$, we have

$$\sin i = \frac{a\mu_0 \sin z}{(a+x)\mu} = \frac{(1-s)\,\mu_0 \sin z}{\mu}$$

whence

$$\tan i = \frac{\sin i}{\sqrt{(1-\sin^2 i)}} = \frac{(1-s)\sin z}{\sqrt{\left[\frac{\mu^2}{\mu_0^2} - (1-s)^2 \sin^2 z\right]}}$$

$$= \frac{(1-s)\sin z}{\sqrt{\left[\cos^2 z - \left(1-\frac{\mu^2}{\mu_0^2}\right) + (2s - s^2)\sin^2 z\right]}}$$

From the equation $\mu^2 = 1 + 4k\delta$ we deduce

$$\frac{d\mu}{\mu} = \frac{2kd\delta}{1 + 4k\delta}$$

and if we introduce as a constant the quantity

$$a = \frac{2k\delta_0}{1 + 4k\delta_0} \qquad (178)$$

(which for Barom. $0^m.76$ and Therm. $0°$ C. is $a = 0.000294211$)

$$\frac{d\mu}{\mu} = \frac{a\,\dfrac{d\delta}{\delta_0}}{1 - 2a\left(1 - \dfrac{\delta}{\delta_0}\right)}$$

We might neglect the square of k, and consequently, also, that of

$$e^{-\beta s} = e^{-\beta s'} - \frac{\alpha\beta}{\sin^2 z} \left[1 - e^{-\beta s'}\right] e^{-\beta s'}$$

$$-\frac{\alpha^2\beta}{1.2\sin^4 z} D\left[(1 - e^{-\beta s'})^2 e^{-\beta s'}\right]$$

$$-\frac{\alpha^3\beta}{1.2.3\sin^6 z} D^2\left[(1 - e^{-\beta s'})^3 e^{-\beta s'}\right]$$

$$. \qquad . \qquad . \qquad . \qquad . \qquad .$$

$$-\frac{\alpha^n\beta}{1.2.3\ldots n\sin^{2n} z} D^{n-1}\left[(1 - e^{-\beta s'})^n e^{-\beta s'}\right]$$

$$- \text{ \&c.} \tag{182}$$

But we have in the numerator of (181)

$$\beta e^{-\beta s} \, ds = - \, de^{-\beta s}$$

and hence, differentiating (182) and substituting the result in (181), we find

$$dr = \frac{\alpha\beta\sin z \, ds'}{(1 - \alpha)\left[\cos^2 z + 2s'\sin^2 z\right]^{\frac{1}{2}}} \Big\{ e^{-\beta s'} + \frac{\alpha}{\sin^2 z} D\left[(1 - e^{-\beta s'})e^{-\beta s'}\right]$$

$$+ \frac{\alpha^2}{1.2\sin^4 z} D^2\left[(1 - e^{-\beta s'})^2 e^{-\beta s'}\right]$$

$$. \qquad . \qquad . \qquad . \qquad .$$

$$+ \frac{\alpha^n}{1.2.3\ldots n\sin^{2n} z} D^n\left[(1 - e^{-\beta s'})^n e^{-\beta s'}\right]$$

$$+ \text{ \&c.} \Big\} \tag{183}$$

To effect the differentiations expressed in the several terms of this series, we take the general expression

$$(1 - e^{-\beta s'})^n e^{-\beta s'} = (- e^{-\beta s'} + 1)^n e^{-\beta s'}$$

$$= \pm\left(e^{-(n+1)\beta s'} - n e^{-n\beta s'} + \frac{n(n-1)}{1.2} e^{-(n-1)\beta s'} - \text{\&c.} \right)$$

where the upper sign is to be used when n is even, and the lower sign when n is odd. Differentiating this n times successively, we have

$$D^n\left[(1 - e^{-\beta s'})e^{-\beta s'}\right] = +\beta^n\left[(n+1)^n e^{-(n+1)\beta s'} - n^n . n e^{-n\beta s'} + \text{\&c.}\right]$$

by means of which, making $n = 1.2.3\ldots$ successively, we reduce (183) to the following form:

$$dr = \frac{\alpha\beta \sin z \, ds'}{(1-\alpha)[\cos^2 z + 2 \, s' \sin^2 z]^{\frac{1}{2}}} \left\{ e^{-\beta s'} + \frac{\alpha\beta}{\sin^2 z}\left(2 e^{-2\beta s'} - e^{-\beta s'} \right) \right.$$

$$+ \frac{\alpha^2 \beta^2}{1.2 \sin^4 z}\left(3^2 e^{-3\beta s'} - 2^2 . 2 e^{-2\beta s'} + e^{-\beta s'} \right)$$

$$+ \frac{\alpha^3 \beta^3}{1.2.3 \sin^6 z}\left(4^3 e^{-4\beta s'} - 3^3 . 3 e^{-3\beta s'} + 2^3 . 3 e^{-2\beta s'} - e^{-\beta s'} \right)$$

$$\left. + \&c. \vphantom{\frac{1}{1}} \right\} \tag{184}$$

We have now to integrate the terms of this series, after having multiplied each by the factor without the brackets. The integrals are to be taken from the surface of the earth, where $s = 0$, to the upper limit of the atmosphere; that is, q being the normal to any stratum (Art. 108), they are to be taken between the limits $q = a$ and $q = a + H$, H being the height of the atmosphere. Now, this height is not known; but since at the upper limit the density is zero and beyond this limit the ray suffers no refraction to infinity, we can without error take the integrals between the limits $q = a$ and $q = \infty$, i.e. between $s = 0$ and $s = 1$. But we may make the upper limit of s also equal to infinity. For, by (176), β will not differ greatly from $\frac{a}{l}$, and consequently will be a very large number, nearly equal to 800, as we find from (167); hence for $s = 1$ we have in (172) $\delta = \dfrac{\delta_0}{(2.718\ldots)^{800}}$ which will be sensibly equal to zero, and consequently the same as we should find by taking $s = \infty$. Hence the integrals may be taken between the limits $s = 0$ and $s = \infty$; consequently, also, according to (180), between the limits $s' = 0$ and $s' = \infty$.

Now, as every term of the series will be of the form

$$\frac{\beta \sin z \, ds' e^{-n\beta s'}}{[\cos^2 z + 2 \, s' \sin^2 z]^{\frac{1}{2}}} = \frac{\beta \, ds' e^{-n\beta s'}}{[\cot^2 z + 2 \, s']^{\frac{1}{2}}} \tag{185}$$

multiplied by constants, we have only to integrate this general form. Let t be a new variable, such that

$$\cot^2 z + 2 s' = \frac{2 t^2}{n\beta} \tag{186}$$

can also be written as follows:*

$$r = \frac{a\sqrt{2\beta}}{1-a}\begin{cases} e^{-\frac{a\beta}{\sin^2 z}} \;\psi(1) \\[2mm] + \; 2^{\frac{1}{2}} \cdot \frac{a\,\beta}{\sin^2 z}\; e^{-\frac{2a\beta}{\sin^2 z}} \;\psi(2) \\[2mm] + \; \frac{3^{\frac{3}{2}}}{1\cdot2} \cdot \frac{a^2\beta^2}{\sin^4 z}\; e^{-\frac{3a\beta}{\sin^2 z}} \;\psi(3) \\[2mm] + \; \frac{4^{\frac{5}{2}}}{1\cdot2\cdot3} \cdot \frac{a^3\beta^3}{\sin^6 z}\; e^{-\frac{4a\beta}{\sin^2 z}} \;\psi(4) \\[2mm] + \; \&c. \end{cases} \tag{191}$$

113. The only remaining difficulty is to determine the function $\psi(n)$, (188). In the case of the horizontal refraction, where $\cot z = 0$ and therefore also $T = 0$, this function becomes independent of (n), and reduces to the well-known integral†

$$\int_0^\infty dt\, e^{-tt} = \frac{\sqrt{\pi}}{2} \tag{192}$$

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† This useful definite integral may be readily obtained as follows. Put $k = \int_0^\infty dt\, e^{-tt}$. Then, since the definite integral is independent of the variable, we also have $k = \int_0^\infty dv\, e^{-vv}$, and, multiplying these expressions together,

$$k^2 = \int_0^\infty dt\, e^{-tt} \int_0^\infty dv\, e^{-vv} = \int_0^\infty \int_0^\infty dt\, dv\, e^{-(tt+vv)}$$

the order of integration being arbitrary. Let

$$v = tu;\; \text{whence } dv = t\,du$$

(for in integrating, regarding v as variable, t is regarded as constant): then we have

$$k^2 = \int_0^\infty \int_0^\infty du\cdot dt\cdot te^{-tt(1+uu)} = \int_0^\infty du \int_0^\infty dt\cdot te^{-tt(1+uu)}$$

$$= \int_0^\infty du\cdot \frac{1}{2(1+u^2)} = \tfrac{1}{2}(\tan^{-1}\infty - \tan^{-1}0) = \frac{\pi}{4}$$

whence

$$k = \int_0^\infty dt\, e^{-tt} = \frac{\sqrt{\pi}}{2}$$

where $\pi = 3.1415926\ldots$. The expression for the horizontal refraction is therefore found at once by putting $\frac{1}{2}\sqrt{\pi}$ for ψ (n) in every term of (191), and $\sin z = 1$, namely:

$$r_0 = \frac{\alpha}{1-\alpha}\sqrt{\frac{\beta\pi}{2}}\left\{\begin{array}{l} e^{-\alpha\beta} \\[2mm] + 2^{\frac{1}{2}}\alpha\beta\, e^{-2\alpha\beta} \\[2mm] + \dfrac{3^{\frac{3}{2}}}{1\,.\,2}\alpha^2\beta^2 e^{-3\alpha\beta} \\[2mm] + \dfrac{4^{\frac{5}{2}}}{1\,.\,2\,.\,3}\alpha^3\beta^3 e^{-4\alpha\beta} \\[2mm] + \&\mathrm{c}. \end{array}\right\} \tag{193}$$

For small values of T, that is, for great zenith distances, we may obtain the value of the integral in (188) by a series of ascending powers of T. We have

$$\int_T^\infty dt\, e^{-tt} = \int_0^\infty dt\, e^{-tt} - \int_0^T dt\, e^{-tt} \tag{194}$$

The first integral of the second member is given by (192). The second is

$$\int_0^T dt\, e^{-tt} = \int_0^T dt\left(1 - t^2 + \frac{t^4}{1\,.\,2} - \frac{t^6}{1\,.\,2\,.\,3} + \&\mathrm{c}.\right)$$

$$= T - \frac{T^3}{3} + \frac{1}{1\,.\,2}\cdot\frac{T^5}{5} - \frac{1}{1\,.\,2\,.\,3}\cdot\frac{T^7}{7} + \&\mathrm{c}. \tag{195}$$

Another development for the same case is obtained by the successive application of the method of integration by parts, as follows:*

$$\int dt\, e^{-tt} = t\, e^{-tt} + 2\int t^2 dt\, e^{-tt}$$

$$= t\, e^{-tt} + \frac{2}{3}t^3 e^{-tt} + \frac{2^2}{3}\int t^4 dt\, e^{-tt}$$

---

* By the formula $\int x\, dy = xy - \int y\, dx$, making always $x = e^{-tt}$, and dy successively $= dt$, $t^2 dt$, $t^4 dt$, &c.

$$= e^{-tt} t \left(1 + \frac{2 t^2}{3} + \frac{(2 t^2)^2}{3 \cdot 5} + \frac{(2 t^2)^3}{3 \cdot 5 \cdot 7} + \&c. \right)$$

whence, by introducing the limits,

$$\int_0^T dt \, e^{-tt} = e^{-TT} \, T \left(1 + \frac{2 \, T^2}{3} + \frac{(2 \, T^2)^2}{3 \cdot 5} + \frac{(2 \, T^2)^3}{3 \cdot 5 \cdot 7} + \&c. \right) \quad (196)$$

As the denominators increase, these series finally become convergent for all values of T; but they are convenient only for small values.

For the greater values of T, a development according to the descending powers may be obtained, also by the method of integration by parts, as follows:* We have

$$\int dt \, e^{-tt} = -\frac{1}{2t} e^{-tt} - \frac{1}{2} \int \frac{dt}{t^2} e^{-tt}$$

$$= -\frac{1}{2t} e^{-tt} + \frac{1}{2^2 t^3} e^{-tt} + \frac{1 \cdot 3}{2^2} \int \frac{dt}{t^4} e^{-tt}$$

Hence

$$\int_T^\infty dt \, e^{-tt} = \frac{e^{-TT}}{2 \, T} \left\{ 1 - \frac{1}{2 \, T^2} + \frac{1 \cdot 3}{(2 \, T^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2 \, T^2)^3} + \cdots \right.$$

$$\left. \pm \frac{1 \cdot 3 \cdot 5 \ldots (2 \, n - 1)}{(2 \, T^2)^n} \right\} \mp \frac{1 \cdot 3 \cdot 5 \ldots (2 \, n + 1)}{2^{n+1}} \int_T^\infty \frac{dt}{t^{2n+2}} e^{-tt} \quad (197)$$

The sum of a number of consecutive terms of this series is alternately greater and less than the value of the integral. But since the factors of the numerators increase, the series will at last become divergent for any value of T. Nevertheless, if we stop at any term, *the sum of all the remaining terms will be less than this term;* for if we take the sum of all the terms in the brackets, the sum of the remaining terms is

$$\pm \frac{1 \cdot 3 \cdot 5 \ldots (2 \, n + 1)}{2^{n+1}} \int_T^\infty \frac{dt}{t^{2n+2}} e^{-tt}$$

---

* By the formula $\int x \, dy = xy - \int y \, dx$, making always $dy = t \, dt \, e^{-tt}$, and x successively $= \frac{1}{t}, \frac{1}{t^3}, \frac{1}{t^5}$, &c.

$$\frac{a_n}{b_n} = \frac{a_{n-1} + (n-1)\, k a_{n-2}}{b_{n-1} + (n-1)\, k b_{n-2}}$$

By the preceding methods, then, the function $\psi(n)$ can be computed for any value of T. A table containing the logarithm of this function for all values of T from 0 to 10, is given by BESSEL (*Fundamenta Astronomiæ*, pp. 36, 37), being an extension of that first constructed by KRAMP. By the aid of this table the computation of the refraction is greatly facilitated.

114. Let us now examine the second term of (179.) This term will have its greatest value in the horizontal refraction, when $z = 90°$, in which case it reduces to

$$-\frac{\alpha \beta e^{-\beta s}\, s\, ds\left[\frac{3}{2}s - 2\alpha\left(1 - e^{-\beta s}\right)\right]}{(1-\alpha)\left[2s - 2\alpha\left(1 - e^{-\beta s}\right)\right]^{\frac{3}{2}}}.$$

Moreover, the most sensible part of the integral corresponds to small values of s, and therefore, since α is also very small, we may put $2\alpha\left(1 - e^{-\beta s}\right) = 2\alpha\beta s$. The integral thus becomes

$$-\frac{\alpha\beta\,(3 - 4\alpha\beta)}{2^{\frac{5}{2}}\,(1-\alpha)\,(1-\alpha\beta)^{\frac{3}{2}}} \int_0^{\infty} s^{\frac{1}{2}}\, ds\, e^{-\beta s}$$

Now we have, by integrating by parts,

$$\int s^{\frac{1}{2}}\, ds\, e^{-\beta s} = -\frac{s^{\frac{1}{2}} e^{-\beta s}}{\beta} + \frac{1}{2\beta}\int s^{-\frac{1}{2}}\, ds\, e^{-\beta s}$$

and hence

$$\int_0^{\infty} s^{\frac{1}{2}}\, ds\, e^{-\beta s} = \frac{1}{2\beta}\int_0^{\infty} s^{-\frac{1}{2}}\, ds\, e^{-\beta s}$$

Putting $\beta s = x^2$, this becomes, by (192),

$$\frac{1}{\beta^{\frac{3}{2}}}\int_0^{\infty} dx\, e^{-xx} = \frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}}$$

Hence the term becomes

$$-\frac{\alpha\,(3 - 4\alpha\beta)}{8\,(1-\alpha)\,(1-\alpha\beta)^{\frac{3}{2}}}\sqrt{\frac{\pi}{2\beta}}$$

Taking BESSEL's value of $h = 116865.8$ toises* $= 227775.7$ metres, and the value of $l = 7993.15$ metres (p. 141), we find by (176) $\beta = 768.57$. Substituting this and $\alpha = 0.000294211$ (p. 146), the value of the above expression, reduced to seconds of arc by dividing by sin $1''$, is found to be only $0''.72$, which in the horizontal refraction is insignificant. This term, therefore, can be neglected (and consequently also all the subsequent terms), and the formula (191) may be regarded as the rigorous expression of the refraction.

115. In order to compute the refraction by (191), it only remains to determine the constants α and β. The constant α might be found from (178) by employing the value of k determined by BIOT by direct experiment upon the refractive power of atmospheric air, but in order that the formula may represent as nearly as possible the observed refractions, BESSEL preferred to determine both α and β from observations.†

Now, α depends upon the density of the air at the place of observation, and is, therefore, a function of the pressure and temperature; and β, which involves l, also depends upon the thermometer, since by the definition of l it must vary with the temperature. The constants must, then, be determined for some assumed normal state of the air, and we must have the means of deducing their values for any other given state. Let

$p_0 =$ the assumed normal pressure,
$\tau_0 = $ " " temperature,
$p = $ the observed pressure,
$\tau = $ " " temperature,
$\delta_0 = $ the normal density corresponding to p_0 and τ_0,
$\delta = $ the density corresponding to p and τ;

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* *Fundamenta Astronomiæ*, p. 40.

† It should be observed that the assumed expression of the density (177) may represent various hypotheses, according to the form given to β. Thus, if we put $\beta = \frac{a}{l}$, we have the form (172) which expresses the hypothesis of a uniform temperature. We may therefore readily examine how far that hypothesis is in error in the horizontal refraction; for by taking the reciprocal of (167) we have in this case $\beta = 796.53$, and hence with $a = 0.00029411$ we find, by taking fifteen terms of the series (193), $r_0 = 39' 54''.5$, which corresponds to Barom. $0^m. 76$, and Therm. $0°$ C. This is $2' 23''.5$ *greater* than the value given by ARGELANDER's Observations (p. 141). Our first hypothesis gave a result too small by more than $7'$, and hence a true hypothesis must be intermediate between these, as we have already shown from a con-

then we have by (171)

$$\delta = \frac{\delta_0}{1 + \epsilon\,(\tau - \tau_0)} \cdot \frac{p}{p_0}$$

in which ϵ is the coefficient of expansion of atmospheric air, or the expansion for 1° of the thermometer. *If the thermometer is Centigrade*, we have, according to BESSEL,*

$$\epsilon = 0.0036438$$

From (178) it follows that α is sensibly proportional to the density, and hence if we put

$$\alpha_0 = \text{the value of } \alpha \text{ for the normal density } \delta_0,$$

we have, for any given state of the air,

$$\alpha = \frac{\alpha_0}{1 + \epsilon(\tau - \tau_0)} \cdot \frac{p}{p_0} \qquad (205)$$

in which for p and p_0 we may use the heights of the barometric column, provided these heights are reduced to the same temperature of the mercury and of the scales.

Again, if

$$l_0 = \text{the height of a homogeneous atmosphere of the temperature}$$
$$\tau_0 \text{ and any given pressure and density,}$$

then the height l for the same pressure and density, when the temperature is τ, is

$$l = l_0\,[1 + \epsilon(\tau - \tau_0)] \qquad (206)$$

The normal state of the air adopted by BESSEL in the determination of the constants, so as to represent BRADLEY's observations, made at the Greenwich Observatory in the years 1750–1762, was a mean state corresponding to the barometer 29.6 inches, and thermometer 50° Fahrenheit = 10° Centigrade; and for this state he found

$$\alpha_0 = 0.000278953$$

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sideration of the law of temperatures. At the same time, we see that the hypothesis of a uniform temperature is nearer to the truth than the first hypothesis, and we are so far justified in adhering to the form $\delta = \delta_0 e^{-\beta s}$ with the modification of substituting a corrected value of β.

* This value, determined by BESSEL, from the observations of stars, differs slightly from the value $\frac{1}{273}$ more recently determined by RUDBERG and REGNAULT by direct experiments upon the refractive power of the air.

or, dividing by sin $1''$,

$$\alpha_0 = 57''.538$$

and

$$h = 116865.8 \text{ toises} = 227775.7 \text{ metres.}$$

For the constant l_0 at the normal temperature 50° F., BESSEL employed

$$l_0 = 4226.05 \text{ toises} = 8236.73 \text{ metres.}*$$

Since the strata of the atmosphere are supposed to be parallel to the earth's surface, BESSEL employed for a the radius of curvature of the meridian for the latitude of Greenwich (the observations of Bradley being taken in the meridian), and, in accordance with the compression of the earth assumed at the time when this investigation was made, he took

$$a = 6372970 \text{ metres.}$$

Hence we have

$$\beta_0 = \frac{h - l_0}{h} \cdot \frac{a}{l_0} = 745.747$$

These values of α_0 and β_0 being substituted for α and β in (193), the horizontal refraction is found to be only about $1'$ too great, which is hardly greater than the probable error of the observed horizontal refraction. At zenith distances less than 85°, however, BESSEL afterwards found that the refraction computed with these values of the constants required to be multiplied by the factor 1.003282 in order to represent the Königsberg observations.

116. By the preceding formulæ, then, the values of the constants α and β can be found for any state of the air, as given by the barometer and thermometer at the place of observation, and then the true refraction might be directly computed by (191). But, as this computation would be too troublesome in practice, the *mean refraction* is computed for the assumed normal values of α and β, and given in the refraction tables. From this mean

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* According to the later determination of REGNAULT which we have used on p. 143, we should have $l_0 = 8286.1$ metres. The difference does not affect the value of BESSEL's tables, which are constructed to represent actual observations.

refraction we must deduce the true refraction in any case by applying proper corrections depending upon the observed state of the barometer and thermometer. For facility of logarithmic computation, BESSEL adopted the form

$$r = r_0 \left(\frac{p}{p_0}\right)^A \left\{\frac{1}{1 + \epsilon\,(\tau - \tau_0)}\right\}^\lambda \tag{207}$$

in which r_0 is the tabular refraction corresponding to p_0 and τ_0, and r is the refraction corresponding to the observed p and τ. Let us see what interpretation must be given to the exponents A and λ. If the pressure remained p_0, the refraction corresponding to the temperature τ would be

$$r_0 + \frac{dr}{d\tau}\,(\tau - \tau_0) + \frac{d^2r}{d\tau^2}\frac{(\tau - \tau_0)^2}{1.2} + \&c.$$

or, with sufficient precision,

$$r_0 \left\{1 + \frac{1}{r_0}\cdot\frac{dr}{d\tau}\,(\tau - \tau_0)\right\}$$

In like manner, if the temperature were constant, and the pressure is increased by the quantity $p - p_0$, the refraction would become nearly

$$r_0 \left\{1 + \frac{1}{r_0}\cdot\frac{dr}{dp}\,(p - p_0)\right\}$$

Hence, when both pressure and temperature vary, we shall have, very nearly,

$$r = r_0 \left\{1 + \frac{1}{r_0}\cdot\frac{dr}{dp}\,(p - p_0)\right\} \times \left\{1 + \frac{1}{r_0}\cdot\frac{dr}{d\tau}\,(\tau - \tau_0)\right\} \tag{208}$$

Now, putting $\dfrac{p}{p_0}$ in (207) under the form $1 + \dfrac{p - p_0}{p_0}$, and developing by the binomial theorem, we have

$$r = r_0 \left\{1 + \frac{A}{p_0}\,(p - p_0) + \&c.\right\} \times \left\{1 - \lambda\epsilon\,(\tau - \tau_0) + \&c.\right\}$$

Therefore, neglecting the smaller terms, we must have

$$A = \frac{p_0}{r_0}\cdot\frac{dr}{dp} \qquad \lambda = -\frac{1}{2\,r_0}\cdot\frac{dr}{d\tau} \tag{209}$$

to determine which we are now to find the derivatives of (191) relatively to p and τ. Put

$$x = \frac{a\beta}{\sin^2 z} \tag{210}$$

and $q_1 = \psi(1)$, $q_2 = 2^{\frac{1}{2}}\psi(2)$, $q_3 = 3^{\frac{1}{2}}\psi(3)$, &c., or in general

$$q_n = n^{\frac{n-1}{2}} \psi(n) \tag{211}$$

then, if we also put

$$Q = x e^{-z} q_1 + \frac{x^2}{1 \cdot 2} e^{-2z} q_2 \ldots\ldots + \frac{x^n}{1 \cdot 2 \ldots\ldots n} e^{-nz} q_n + \&c. \tag{212}$$

the formula (191) becomes

$$(1 - a) r = \sin^2 z \sqrt{\frac{2}{\beta}} \cdot Q \tag{213}$$

in which, since the variations of $\frac{a}{1-a}$ in (191) are sensibly the same as those of a, we may regard $1-a$ as constant. Differentiating this, observing that Q varies with both p and τ, while β varies only with τ, we have

$$\left.\begin{aligned} (1 - a) \frac{dr}{dp} &= \sin^2 z \sqrt{\frac{2}{\beta}} \cdot \frac{dQ}{dp} \\ (1 - a) \frac{dr}{d\tau} &= \sin^2 z \sqrt{\frac{2}{\beta}} \cdot \frac{dQ}{d\tau} - (1 - a)\frac{r}{2\beta} \cdot \frac{d\beta}{d\tau} \end{aligned}\right\} \tag{214}$$

In differentiating Q, it will be convenient to regard it as a function of the two variables x and β, the quantities q_1, q_2, &c. varying only with β. We have, since β does not vary with p,

$$\frac{dQ}{dp} = \frac{dQ}{dx} \cdot \frac{dx}{dp} \tag{215}$$

and since both x and β vary with τ,

$$\frac{dQ}{d\tau} = \frac{dQ}{dx} \cdot \frac{dx}{d\tau} + \frac{dQ}{d\beta} \cdot \frac{d\beta}{d\tau} \tag{216}$$

From (212) we find

$$\frac{dQ}{dx} = \frac{1-x}{x} \, Q' \tag{217}$$

in which

$$Q' = x\,e^{-x}\, q_1 + \frac{x^2}{1.2}\, e^{-2x}\, 2q_2 + \frac{x^3}{1.2.3}\, e^{-3x}\, 3q_3 + \&c. \tag{218}$$

Also,

$$\frac{dQ}{d\beta} = x\,e^{-x}\, \frac{dq_1}{d\beta} + \frac{x^2}{1.2}\, e^{-2x}\, \frac{dq_2}{d\beta} + \&c. \tag{219}$$

in which we have generally, by (211),

$$\frac{dq_n}{d\beta} = n^{\frac{2n-1}{2}}\, \frac{d\psi(n)}{dT} \cdot \frac{dT}{d\beta}$$

But by (200), in which $u_0 = \psi(n)$, we have

$$n^{\frac{2n-1}{2}}\, \frac{d\psi(n)}{dT} = 2T\, q_n = -n^{\frac{2n-1}{2}}$$

and by (187)

$$\frac{dT}{d\beta} = \frac{T}{2\beta}$$

whence

$$\frac{dq_n}{d\beta} = \frac{T^2}{\beta}\, q_n - \frac{T}{2\beta}\, n^{\frac{2n-1}{2}}$$

$$= \frac{\cot^2 z}{2}\, n\,q_n - \frac{\cot z}{2\sqrt{2\beta}}\, n^n$$

Substituting the values of this expression for $n = 1, 2, 3,$ &c. in (219), we have

$$\frac{dQ}{d\beta} = \frac{\cot^2 z}{2}\left(x\,e^{-x}\, q_1 + \frac{x^2}{1.2}\, e^{-2x}\, 2q_2 + \frac{x^3}{1.2.3}\, e^{-3x}\, 3q_3 + \&c. \right)$$

$$- \frac{\cot z}{2\sqrt{2\beta}}\left(x\,e^{-x} + \frac{x^2}{1.2}\, e^{-2x}\, 2^2 + \frac{x^3}{1.2.3}\, e^{-3x}\, 3^3 + \&c. \right)$$

The first series in this expression $= Q'$. The second, when $e^{-x},\ e^{-2x},$ &c. are developed in series, becomes

$$x + x^2 + x^3 + \&c. = \frac{x}{1-x}$$

The true refraction then takes the form

$$r = \alpha \beta^A \gamma^\lambda \tan z \qquad\qquad (223)$$

The quantity here denoted by β is the ratio of the observed and normal heights of the barometer, both being reduced to the same temperature of the mercury and of their scales. First, to correct for the temperature of the scale, let $b^{(l)}$, $b^{(e)}$, or $b^{(m)}$ denote the observed reading of the barometer scale according as it is graduated in Paris lines, English inches, or French metres. The standard temperatures of the Paris line is 13° Réaumur, of the English inch 62° Fahrenheit, and of the French metre 0° Centigrade; that is, the graduations of the several scales indicate true heights only when the attached thermometers indicate these temperatures respectively. The expansion of brass from the freezing point to the boiling point is .0018782 of its length at the freezing point. If then the reading of the attached thermometer is denoted either by r', f', or c', according as it is Réaumur's, Fahrenheit's, or the Centigrade, the true height observed will be (putting $s = 0.0018782$)

$$b^{(l)} \cdot \frac{1 + \frac{s}{80} r'}{1 + \frac{s}{80} \cdot 13}, \qquad b^{(e)} \cdot \frac{1 + \frac{s}{180}(f' - 32)}{1 + \frac{s}{180} \cdot 30}, \qquad b^{(m)} \cdot \frac{1 + \frac{s}{100} \cdot c'}{1}$$

or

$$b^{(l)} \cdot \frac{80 + r's}{80 + 13s}, \qquad b^{(e)} \cdot \frac{180 + (f' - 32)s}{180 + 30s}, \qquad b^{(m)} \cdot \frac{100 + c's}{100} \qquad (224)$$

where the multipliers $1 + \frac{s}{80} r'$, &c. evidently reduce the reading to what it would have been if the observed temperature had been that of freezing, and the divisors $1 + \frac{s}{80} \cdot 13$, &c. further reduce these to the respective temperatures of graduation, and consequently give the true heights.

This true height of the mercury will be proportional to the pressure only when the temperature of the mercury is constant. We must, therefore, reduce the height to what it would be if the temperature were equal to the adopted normal temperature, which is in our table 8° Réaumur = 50° F. = 10° C. Now, mercury expands $\frac{1}{55.5}$ of its volume at the freezing point of water, when

its temperature is raised from that point to the boiling point of water. Hence, putting $q = \frac{1}{55.5}$, the above heights will be reduced to the normal temperature by multiplying them respectively by the factors

$$\frac{80 + 8q}{80 + r'q}, \quad \frac{180 + 18q}{180 + (f' - 32)q}, \quad \frac{100 + 10q}{100 + c'q} \quad (225)$$

The normal height of the barometer adopted by BESSEL was 29.6 inches of Bradley's instrument, or 333.28 Paris lines; but it afterwards appeared that this instrument gave the heights too small by $\frac{1}{2}$ a Paris line, so that the normal height in the tables is 333.78 Paris lines, at the adopted normal temperature of 8° R. Reducing this to the standard temperature of the Paris line = 13° R., we have

$$b_0 = 333.78 \frac{80 + 8s}{80 + 13s} \quad (226)$$

In comparing this with the observed heights, the $b^{(e)}$ and $b^{(m)}$ must be reduced to lines by observing that one English inch = 11.2595 Paris lines, and one metre = 443.296 Paris lines. Making this reduction, the value of $\beta = \frac{p}{p_0}$ is found by dividing the product of (224) and (225) by (226). The result may then be separated into two factors, one of which depends upon the observed height of the barometric column, and the other upon the attached thermometer; so that if we put

$$
\begin{aligned}
B &= \frac{b^{(l)}}{333.78} \cdot \frac{80 + 8q}{80 + 8s} \\
&= b^{(e)} \cdot \frac{11.2595}{333.78} \cdot \frac{80 + 13s}{80 + 8s} \cdot \frac{180 + 18q}{180 + 30s} \\
&= b^{(m)} \cdot \frac{443.296}{333.78} \cdot \frac{80 + 13s}{80 + 8s} \cdot \frac{100 + 10q}{100}
\end{aligned}
\quad (227)
$$

and

$$T = \frac{80 + r's}{80 + r'q} = \frac{180 + (f' - 32)s}{180 + (f' - 32)q} = \frac{100 + c's}{100 + c'q}$$

we shall have $\beta = BT$, or

$$\log \beta = \log B + \log T \quad (228)$$

The quantity γ would be computed directly under the form

$$\gamma = \frac{1}{1 + \epsilon(\tau - \tau_0)}$$

if τ_0 were at once the freezing point and the normal temperature of the tables; for ϵ is properly the expansion of the air for each degree of the thermometer above the freezing point, the density of the air at this point being taken as the unit of density. But if the normal temperature is denoted by τ_0, that of the freezing point by τ_1, the observed by τ, we shall have

$$\gamma = \frac{1 + \epsilon(\tau_0 - \tau_1)}{1 + \epsilon(\tau - \tau_1)}$$

an expression which, if we neglect the square of ϵ, will be reduced to the above more simple one by dividing the numerator and denominator by $1 + \epsilon(\tau_0 - \tau_1)$. BESSEL adopted for τ_0 the value 50° F. by BRADLEY's thermometer; but as this thermometer was found to give 1°.25 too much, the normal value of the tables is $\tau_0 = 48°.75$ F. Hence, if r, f, or c denote the temperature indicated by the external thermometer, according as it is Réaumur, Fahr., or Cent., we have*

$$\left.\begin{array}{l}\gamma = \dfrac{180 + 16.75 \times 0.36438}{180 + \frac{9}{4}r \times 0.36438} \\[2mm] = \dfrac{180 + 16.75 \times 0.36438}{180 + (f - 32) \times 0.36438} \\[2mm] = \dfrac{180 + 16.75 \times 0.36438}{180 + \frac{9}{5}c \times 0.36438}\end{array}\right\} \quad (229)$$

The tables constructed according to these formulæ give the values of log B, log T, and log γ, with the arguments barometer, attached thermometer, and external thermometer respectively, and the computation of the true refraction is rendered extremely simple. An example has already been given in Art. 107.

118. In the preceding discussion we have omitted any consideration of the hygrometric state of the atmosphere. The

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* *Tabulæ Regiomontanæ*, p. LXII.

refractive power of aqueous vapor is greater than that of atmospheric air of the same density, but under the same pressure its density is less than that of air; and LAPLACE has shown that "the increase of the greater refractive power of vapor is in a great degree compensated by its decrease of density."*

119. *Refraction table with the argument true zenith distance.*—When the true zenith distance ζ is given, we may still find the refraction from the usual tables, or Col. A of Table II., where the apparent zenith distance z is the argument, by successive approximations. For, entering the table with ζ instead of z, we shall obtain an approximate value of r, which, subtracted from ζ, will give an approximate value of z; with this a more exact value of r can be found, and a second value of z, and so on, until the computed values of r and z exactly satisfy the equation $z = \zeta - r$. But it is more convenient to obtain the refraction directly with the argument ζ. For this purpose Col. B of Table II. gives the quantities α', A', λ', which are entirely analogous to the α, A, and λ, so that the refraction is computed under the form

$$r = \alpha' \beta^{A'} \gamma^{\lambda'} \tan \zeta \qquad (230)$$

where β and γ have the same values as before.

The values of α', A', and λ' are deduced from those of α, A, and λ after the latter have been tabulated. They are to be so determined as to satisfy the equations

$$\alpha \beta^{A} \gamma^{\lambda} \tan z = \alpha' \beta^{A'} \gamma^{\lambda'} \tan \zeta \qquad (231)$$
$$z = \zeta - \alpha' \beta^{A'} \gamma^{\lambda'} \tan \zeta \qquad (232)$$

and this for any values of β and γ. Let (z) denote the value of z which corresponds to ζ when $\beta = 1$, $\gamma = 1$; that is, when the refraction is at its mean tabular value. The value of (z) may be found by successive approximations from Col. A., as above explained. Let (α), (A), (λ), and (r) denote the corresponding values of α, A, λ, r. We have

$$(r) = (\alpha) \tan (z) = \alpha' \tan \zeta$$
$$(z) = \zeta - \alpha' \tan \zeta$$

whence, by (232),

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* *Méc. Cél.* Book X.

$$z = (z) - a' \tan \zeta \, (\beta^{A'} \gamma^{\lambda'} - 1)$$

But, taking Napierian logarithms, we have

$$l\,(\beta^{A'} \gamma^{\lambda'}) = A'\, l\beta + \lambda'\, l\gamma$$

and hence, e being the Napierian base,

$$\beta^{A'} \gamma^{\lambda'} = e^{A'l\beta + \lambda'l\gamma} = 1 + (A'\, l\beta + \lambda'\, l\gamma) + \&c.$$

where, as β and γ differ but little from unity, the higher powers of $A'l\beta + \lambda'l\gamma$ may be omitted. Hence

$$z = (z) - (r)\,[A'\, l\beta + \lambda'\, l\gamma]$$

Now, taking the logarithm of (231), we have

$$l\,(a \tan z) + A\, l\beta + \lambda\, l\gamma = l\,(a' \tan \zeta) + A'\, l\beta + \lambda'\, l\gamma$$

The first member is a function of z, which we may develop as a function of (z); for, denoting this first member by fz, and putting

$$y = -\,(r)\,[A'\, l\beta + \lambda'\, l\gamma]$$

we have $z = (z) + y$, and hence

$$fz = f\,[(z) + y] = f\,(z) + \frac{df(z)}{d\,(z)}\, y + \&c.,$$

where we may also neglect the higher powers of y. But since $f(z)$ is what fz becomes when $z = (z)$, and consequently $A = (A)$, $\lambda = (\lambda)$, we have

$$f\,(z) = l\,[(a) \tan (z)] + (A)\, l\beta + (\lambda)\, l\gamma$$

$$\frac{df(z)}{d\,(z)} = \frac{dl[(a) \tan (z)]}{d\,(z)} = \frac{d\,[(a) \tan (z)]}{(a) \tan (z)\, d\,(z)} = \frac{1}{(r)} \cdot \frac{d\,(r)}{d\,(z)}$$

Hence we have

$$fz = l\,[(a) \tan (z)] + (A)\, l\beta + (\lambda)\, l\gamma - \frac{d\,(r)}{d\,(z)}\,[A'\, l\beta + \lambda'\, l\gamma]$$

$$= l\,[a' \tan \zeta] + A'\, l\beta + \lambda'\, l\gamma$$

or, since $(a) \tan (z) = a' \tan \zeta$

$$A'\left\{1+\frac{d(r)}{d(z)}\right\}l\beta + \lambda'\left\{1+\frac{d(r)}{d(z)}\right\}l\gamma = (A)\,l\beta + (\lambda)l\gamma$$

Since this is to be satisfied for indeterminate values of β and γ, the coefficients of $l\beta$ and $l\gamma$ in the two members must be equal; and therefore

$$A'=\frac{(A)}{1+\dfrac{d(r)}{d(z)}}\cdot$$

$$\lambda'=\frac{(\lambda)}{1+\dfrac{d(r)}{d(z)}}$$

$\left.\rule{0pt}{4cm}\right\}$ (233)

and also

$$\alpha'=(\alpha)\frac{\tan(z)}{\tan(\zeta)}$$

All the quantities in the second members of these formulæ may be found from Column A of Table II., and thus Column B may be formed.*

If we put

$$k'=\alpha'\,\beta^{A'}\gamma^{\lambda'}$$

we shall now find the refraction under the form

$$r=k'\tan\zeta$$

120. *To find the refraction of a star in right ascension and declination.*

The declination δ and hour angle t of the star being given, together with the latitude φ of the place of observation, we first compute the true zenith distance ζ and the parallactic angle q by (20). The refraction will be expressed under the form

$$r=k'\tan\zeta$$

in which

$$k'=\alpha'\beta^{A'}\gamma^{\lambda'}$$

The latitude and azimuth being here constant (since refraction acts only in the vertical circle), we have from (50), by put-

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* See also BESSEL, *Astronomische Untersuchungen*, Vol. I. p. 159.

ting $d\varphi = 0$, $dA = 0$, $d\zeta = r = k' \tan \zeta$, $dt = - da$, ($a = $ star's right ascension),

$$\left.\begin{array}{r} d\delta = - k' \tan \zeta \cos q \\ \cos \delta\, da = - k' \tan \zeta \sin q \end{array}\right\} \quad (234)$$

which are readily computed, since the logarithms of $\tan \zeta \cos q$ and $\tan \zeta \sin q$ will already have been found in computing ζ by (20). The value of log k' will be found from Table II. Column B, with the argument ζ.

The values of $d\delta$ and da thus found are those which are to be algebraically added to the *apparent* declination and right ascension to free them from the effect of refraction.

The mean value of k' is about $57''$, which may be employed when a very precise result is not required.

DIP OF THE HORIZON.

121. The *dip of the horizon* is the angle of depression of the visible sea horizon below the true horizon, arising from the elevation of the eye of the observer above the level of the sea.

Let CZ, Fig. 17, be the vertical line of an observer at A, whose height above the level of the sea is AB. The plane of the true horizon of the observer at A is a plane at right angles to the vertical line (Art. 3). Let a vertical plane be passed through CZ, and let BTD be the intersection of this plane with the earth's surface regarded as a sphere, AH its intersection with the horizontal plane. Draw ATH' in this plane, tangent to the circular section of the earth at T. Disregarding for the present the effect of the atmosphere, T will be the most distant point of the surface visible from A. If we now conceive the vertical plane to revolve about CZ as an axis, AH will generate the plane of the celestial horizon, while AH' will generate the surface of a cone touching the earth in the small circle called the visible horizon; and the angle HAH' will be the dip of the horizon.
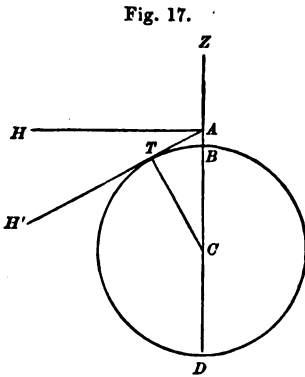
Fig. 17.

122. *To find the dip of the horizon, neglecting the atmospheric refraction.* Let

$x =$ the height of the eye $= AB$,
$a =$ the radius of the earth,
$D =$ the dip of the horizon.

We have in the triangle CAT, $ACT = HAH' = D$, and hence

$$\tan D = \frac{AT}{CT}$$

By geometry, we have

$$AT = \sqrt{AB \times AD} = \sqrt{x\,(2\,a + x)}$$

whence

$$\tan D = \frac{\sqrt{2\,ax + x^2}}{a} = \sqrt{\frac{2\,x}{a} + \left(\frac{x}{a}\right)^2}$$

As x is always very small compared with a, the square of the fraction $\frac{x}{a}$ is altogether inappreciable: so that we may take simply

$$\tan D = \sqrt{\frac{2\,x}{a}} \tag{235}$$

123. *To find the dip of the horizon, having regard to the atmospheric refraction.*

The curved path of a ray of light from the point T, Fig. 18, to the eye at A, is the same as that of a ray from A to T; and this is a portion of the whole path of a ray (as from a star S) which passes through the point A, and is tangent to the earth's surface at T. The direction in which the observer at A sees the point T is that of the tangent to the curved path at A, or AH'; the true dip is therefore the angle HAH', and is less than that found in the preceding article. It is also evident that the most distant visible point of the earth's

Fig. 18.
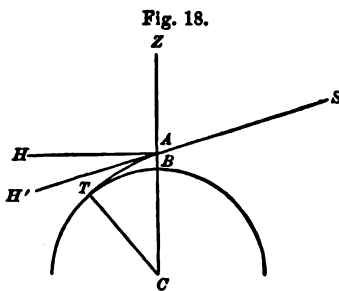
surface is more remote from the observer than it would be if the earth had no atmosphere.

Now, recurring to the investigation of the refraction in **Art.** 108, we observe that the angle HAH' is the complement of the angle of incidence of the ray at the point A, there denoted by i; and it was there shown that if q, μ, and i are respectively the normal, the index of refraction, and the angle of incidence for a point elevated above the earth's surface, while a, μ_0, and z are the same quantities at the surface, we have

$$q \, \mu \, \sin i = a \, \mu_0 \, \sin z$$

But in the present case we have $z = 90°$; and hence, putting

$$D' = \text{the true dip} = 90° - i$$
$$q \;\; = a + x$$

we have

$$\sin i = \cos D' = \frac{\mu_0}{\mu} \cdot \frac{a}{a+x} = \frac{\mu_0}{\mu}\left(1 + \frac{x}{a}\right)^{-1}$$

Developing and neglecting the square of $\dfrac{x}{a}$ as before,

$$\cos D' = \frac{\mu_0}{\mu}\left(1 - \frac{x}{a}\right) \bullet \qquad\qquad (236)$$

which would suffice to determine D' when μ_0 and μ have been obtained from the observed densities of the air at the observer and at the level of the sea. But, as D' is small, it is more convenient to determine it from its sine; and we may also introduce the density of the air directly into the formula by putting (Art. 110),

$$\frac{\mu_0}{\mu} = \sqrt{\frac{1 + 4 k \delta_0}{1 + 4 k \delta}}$$

Substituting the value of a from (178), namely,

$$a = \frac{2 k \delta_0}{1 + 4 k \delta_0}$$

we may give this the form

$$\frac{\mu_0}{\mu} = \left\{ \frac{1}{1 - 2\alpha\left(1 - \frac{\delta}{\delta_0}\right)} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - 2\alpha\left(1 - \frac{\delta}{\delta_0}\right) \right\}^{-\frac{1}{2}}$$

which, by neglecting the square of the second term, gives

$$\frac{\mu_0}{\mu} = 1 + \alpha\left(1 - \frac{\delta}{\delta_0}\right)$$

Hence, still neglecting the higher powers of α and $\frac{x}{a}$, as well as their product, we have

$$\sin D' = \sqrt{1 - \cos^2 D'} = \sqrt{\left\{ \frac{2x}{a} - 2\alpha\left(1 - \frac{\delta}{\delta_0}\right) \right\}} \quad (237)$$

which agrees with the formula given by LAPLACE, *Méc. Cél.* Book X.

For an altitude of a few feet, the difference of pressure will not sensibly affect the value of D', and may be disregarded, especially since a very precise determination of the dip is not possible unless we know the density of the air *at the visible horizon*, which cannot usually be observed. We may, however, assume the temperature of the water to be that of the lowest stratum of the air, and, denoting this by τ_0, while τ denotes the temperature of the air at the height of the eye, we have [making $p = p_0$ in (171)], approximately,

$$\frac{\delta}{\delta_0} = \frac{1}{1 + \epsilon(\tau - \tau_0)} = 1 - \epsilon(\tau - \tau_0)$$

in which for Fahrenheit's thermometer $\epsilon = 0.002024$. Hence

$$\sin D' = \sqrt{\frac{2x}{a}} \left\{ 1 - \frac{a}{x}\alpha\epsilon(\tau - \tau_0) \right\}^{\frac{1}{2}}$$

$$= \sin D \left\{ 1 - \frac{\alpha\epsilon(\tau - \tau_0)}{\sin^2 D} \right\}$$

where D is the dip, computed by (235), when the refraction is neglected, the sine of so small an angle being put for its tangent. If we substitute the values $\alpha = 0.00027895$, $\sin D = D \sin 1''$, and $\epsilon = 0.002024$, this formula becomes

$$D' = D - \frac{24021(\tau - \tau_0)}{D}$$

in which D is in seconds. If D is expressed in minutes in the last term, it will be sufficiently accurate to take

$$D' = D - 400 \times \frac{\tau - \tau_0}{D} \qquad (238)$$

This will give $D' = D$ when $\tau = \tau_0$, as it should do, since in that case the atmosphere is supposed to be of uniform density from the level of the sea to the height of the observer. If $\tau < \tau_0$, we have $D' > D$. In extreme cases, where τ is much greater than τ_0, we may have $D' < 0$, or negative, and the visible horizon will appear above the level of the eye, a phenomenon occasionally observed. I know of no observations sufficiently precise to determine whether this simple formula, deduced from theoretical considerations, accurately represents the observed dip in every case.

124. If, however, we wish to compute the value of D' for a mean state of the atmosphere without reference to the actually observed temperatures, we may proceed as follows: In the equation above found,

$$\cos D' = \frac{\mu_0}{\mu} \cdot \frac{a}{a + x}$$

we may substitute the value

$$\left(\frac{\mu}{\mu_0}\right)^{n+1} = \frac{a}{a + x}$$

which is our first hypothesis as to the law of decrease of density of the strata of the atmosphere, Art. 109. This hypothesis will serve our present purpose, provided n is so determined as to represent the actually observed mean *horizontal* refraction. We have, then,

$$\cos D' = \left(1 + \frac{x}{a}\right)^{-\frac{n}{n+1}}$$

and developing, neglecting the higher powers of $\frac{x}{a}$,

$$\cos D' = 1 - \frac{n}{n+1} \cdot \frac{x}{a}$$

$$\sin D' = \sqrt{\frac{n}{n+1} \cdot \frac{2x}{a}} = \tan D \sqrt{\frac{n}{n+1}}$$

or

$$D' = D \sqrt{\frac{n}{n+1}}$$

To determine n, we have by (160), reducing r_0 to seconds,

$$n = \frac{4 k \delta_0}{(r_0 \sin 1'')^2}$$

where, for Barom. $0^m.76$, Therm. $10°C.$, which nearly represent the mean state of the atmosphere at the surface of the earth, we have $4k\delta_0 = 0.00056795$, and $r_0 = 34'\ 30''$ (which is about the mean of the determinations of the horizontal refraction by different astronomers); and hence we find

$$n = 5.639, \qquad \sqrt{\frac{n}{n+1}} = 0.9216 = 1 - 0.0784$$

$$D' = D - .0784 D \qquad (239)$$

The coefficient .0784 agrees very nearly with DELAMBRE'S value .07876, which was derived from a large number of observations upon the terrestrial refraction at different seasons of the year.

To compute D' directly, we have

$$D' = \frac{0.9216}{\sin 1''} \sqrt{\frac{2}{a}} \times \sqrt{x}$$

If x is in feet, we must take a in feet. Taking the mean value $a = 20888625$ feet, and reducing the constant coefficient of $\sqrt{x}$, we have

$$D' = 58''.82 \sqrt{x} \text{ in feet.} \qquad (240)$$

Table XI., Vol. II., is computed by this formula.

125. To find the distance of the sea horizon, and the distance of an object of known height just visible in the horizon.—The small portion TA, Fig. 19, of the curved path of a ray of
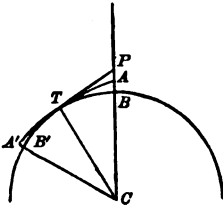
Fig. 19.

light, may be regarded as the arc of a circle; and then the refraction elevates A as seen from T as much as it elevates T as seen from A. Drawing the tangent TP, the observer at T would see the point A at P; and if the chord TA were drawn, the angle PTA would be the refraction of A. This refraction, being the same as that of T as seen from A, is, by (239), equal to $.0784\,D$. In the triangle TPA, TAP is so nearly a right angle (with the small elevations of the eye here considered) that if we put

$$x_1 = AP$$

we may take as a sufficient approximation

$$x_1 = TA \times \tan PTA = a \tan D \times .0784 \tan D$$

But we have $a \tan^2 D = 2x$, and hence

$$x_1 = .1568\,x$$

Putting

$$d = \text{the distance of the sea horizon,}$$

we have

$$PT = \sqrt{(2CB + PB) \times PB}$$

or, nearly,

$$d = \sqrt{2a\,(x + x_1)} = \sqrt{2.3136\,ax}$$

If x is given in feet, we shall find d in statute miles by dividing this value by 5280. Taking a as in the preceding article, we find

$$\frac{\sqrt{2.3136\,a}}{5280} = 1.317$$

and, therefore,

$$d \ \text{(in statute miles)} = 1.317 \sqrt{x \ \text{in feet.}} \tag{241}$$

If an observer at A' at the height $A'B' = x'$ sees the object A, whose height is x, in the horizon, he must be in the curve de-

scribed by the ray from A which touches the earth's surface at T. The distance of A' from T will be $= 1.317 \sqrt{x'}$, and hence the whole distance from A to A' will be $= 1.317 \, (\sqrt{x} + \sqrt{x'})$.

The above is a rather rough approximation, but yet quite as accurate as the nature of the problem requires; for the anomalous variations of the horizontal refraction produce greater errors than those resulting from the formula. By means of this formula the navigator approaching the land may take advantage of the first appearance of a mountain of known height, to determine the position of the ship. For this purpose the formula (241) is tabulated with the argument "height of the object or eye;" and the sum of the two distances given in the table, corresponding to the height of the object and of the eye respectively, is the required distance of the object from the observer.

126. *To find the dip of the sea at a given distance from the observer.* —By the dip of the sea is here understood the apparent depression of any point of the surface of the water nearer than the visible horizon. Let T, Fig. 20, be such a point, and A the position of the observer. Let TA' be a ray of light from T, tangent to the earth's surface at T, meeting the vertical line of the observer in A'. Put
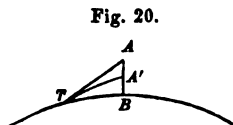
Fig. 20.

$D'' =$ the dip of T as seen from A,
$d \ =$ the distance of T in statute miles,
$x \ =$ the height of the observer's eye in feet $= AB$,
$x' = A'B$.

We have, by (241),

$$x' = \left(\frac{d}{1.317} \right)^2$$

and the dip of T, as seen from A', is, therefore, by (240),

$$= 58''.82 \sqrt{x'} = 44''.66 \, d.$$

Now, supposing the chords TA, TA' to be drawn, the dip of T at A exceeds that at A' by the angle ATA', very nearly; and we have nearly

$$\text{angle } ATA' = \frac{AA'}{TA'} \times \frac{1}{\sin 1''} = \frac{x - x'}{5280 \, d \sin 1''}$$

discussed in connection with the theory of occultations in Chapter X.

The angular semidiameter of a spherical body is the angle subtended at the place of observation by the radius of the disc. I shall here call it simply the semidiameter, and distinguish the linear semidiameter as the *radius*.

Let *O*, Fig. 21, be the centre of the earth, *A* the position of an observer on its surface, *M* the centre of the observed body; *OB*, *AB'*, tangents to its surface, drawn from *O* and *A*. The triangle *OBM* revolved about *OM* as an axis will describe a cone touching the spherical body in the small circle described by the point *B*, and this circle is the *disc* whose angular semidiameter at *O* is *MOB*. Put

Fig. 21.
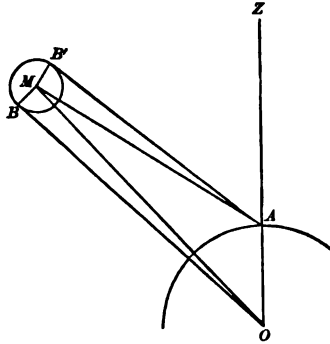
S = the geocentric semidiameter, *MOB*,

S' = the apparent semidiameter, *MAB'*,

Δ, Δ' = the distances of the centre of the body from the centre of the earth and the place of observation respectively,

a = the equatorial radius of the earth,

a' = the radius of the body,

then the right triangles *OMB*, *AMB'* give

$$\sin S = \frac{a'}{\Delta} \qquad \sin S' = \frac{a'}{\Delta'} \qquad (244)$$

But if

π = the equatorial horizontal parallax of the body,

we have, Art. 89,

$$\sin \pi = \frac{a}{\Delta}$$

and hence

$$\sin S = \frac{a'}{a} \sin \pi \qquad \sin S' = \frac{\Delta}{\Delta'} \sin S \qquad (245)$$

or, with sufficient precision in most cases,

$$S = \frac{a'}{a} \pi \qquad S' = \frac{\Delta}{\Delta'} S \qquad (246)$$

The geocentric semidiameter and the horizontal parallax have therefore a constant ratio $= \dfrac{a'}{a}$. For the moon, we have

$$\frac{a'}{a} = 0.272956 \qquad\qquad (247)$$

as derived from the Greenwich observations and adopted by HANSEN (*Tables de la Lune*, p. 39).

If the body is in the horizon of the observer, its distance from him is nearly the same as from the centre of the earth, and hence the geocentric is frequently called the horizontal semidiameter; but this designation is not exact, as the latter is somewhat greater than the former. In the case of the moon the difference is between $0''.1$ and $0''.2$. See Table XII.

If the body is in the zenith, its distance from the observer is less than its geocentric distance by a radius of the earth, and the apparent semidiameter has then its greatest value.

The apparent semidiameter at a given place on the earth's surface is computed by the second equation of (245) or (246), in which the value of $\dfrac{\Delta}{\Delta'}$ is that found by (104); so that, putting $z =$ the true (geocentric) zenith distance of the body, $\zeta' =$ the apparent zenith distance (affected by parallax), $A =$ its azimuth, $\varphi - \varphi'$ the reduction of the latitude, we have, (by (111) and (104),

$$\left. \begin{aligned} \gamma &= (\varphi - \varphi') \cos A \\ \sin S' &= \sin S \, \frac{\sin (\zeta' - \gamma)}{\sin (\zeta - \gamma)} \end{aligned} \right\} \qquad (248)$$

129. This last formula is rigorous, but an approximate formula for computing the difference $S' - S$ will sometimes be convenient. In (103) we may put

$$\frac{\cos (\varphi - \varphi')}{\cos \gamma \cos \frac{1}{2} (\zeta' - \zeta)} = 1$$

without sensible error in computing the very small difference in question; we thus obtain

$$\frac{\Delta'}{\Delta} = 1 - \rho \sin \pi \cos \left[\tfrac{1}{2} (\zeta' + \zeta) - \gamma \right]$$

Putting
$$m = \rho \sin \pi \cos \left[\tfrac{1}{2} (\zeta' + \zeta) - \gamma \right] \qquad (249)$$
we have
$$\frac{\Delta}{\Delta'} = \frac{1}{1-m} = 1 + m + m^2 + \&c.$$

and hence, since the third power of m is evidently insensible,

$$S' - S = Sm + Sm^2 \qquad (250)$$

which is practically as exact as (248). The value of ζ' required in (249) will be found with sufficient accuracy by (114), or

$$\zeta' - \zeta = \rho \pi \sin (\zeta' - \gamma)$$

The quantity $S' - S$ is usually called the *augmentation* of the semidiameter. It is appreciable only in the case of the moon.

130. If we neglect the compression of the earth, which will not involve an error of more than $0''.05$ even for the moon,[*] we may develop (250) as follows. Putting $\rho = 1$ and $\gamma = 0$ in (249), we may take

$$\begin{aligned}
m &= \sin \pi \cos \tfrac{1}{2} (\zeta' + \zeta) \\
&= \sin \pi \cos \left[\zeta' - \tfrac{1}{2} (\zeta' - \zeta) \right] \\
&= \sin \pi \cos \zeta' + \tfrac{1}{2} \sin \pi \sin (\zeta' - \zeta) \sin \zeta \\
&= \sin \pi \cos \zeta' + \tfrac{1}{2} \sin^2 \pi \sin^2 \zeta'
\end{aligned}$$

which substituted in (250) gives, by neglecting powers of $\sin \pi$ above the second,

$$\begin{aligned}
S' - S &= S \sin \pi \cos \zeta' + \tfrac{1}{2} S \sin^2 \pi \sin^2 \zeta' + S \sin^2 \pi \cos^2 \zeta' \\
&= S \sin \pi \cos \zeta' + \tfrac{1}{2} S \sin^2 \pi + \tfrac{1}{2} S \sin^2 \pi \cos^2 \zeta'
\end{aligned}$$

But we have

$$S = \frac{a'}{a} \pi = \frac{a'}{a} \cdot \frac{\sin \pi}{\sin 1''}$$

---

[*] The greatest declination of the moon being less than $30°$, it can reach great altitudes only in low latitudes, where the compression is less sensible. A rigorous investigation of the error produced by neglecting the compression shows that the maximum error is less than $0''.06$.

and if we put

$$h = \frac{a}{a'} \sin 1'', \qquad \log h = 5.2495$$

we have $\sin \pi = hS$, which substituted above gives the following formula for computing the augmentation of the moon's semidiameter:

$$S' - S = h\,S^2 \cos \zeta' + \tfrac{1}{2} h^2\,S^3 + \tfrac{1}{2} h^2\,S^3 \cos^2 \zeta' \qquad (251)$$

EXAMPLE.—Find the augmentation for $\zeta' = 40°$, $S = 16' 0''$ $= 960''$.

log S^2	5.9645	log S^3	8.947	1st term	=	12''.54
log h	5.2495	log $\tfrac{1}{2} h^2$	0.198	2d "	=	0 .14
log cos ζ'	9.8843	log 2d term	9.145	3d "	=	0. 08
log 1st term	1.0983	log cos$^2 \zeta'$	9.769	$S' - S$	=	12 .76
		log 3d term	8.914			

The value of $S' - S$ may be taken directly from Table XII. with the argument apparent altitude $= 90° - \zeta'$.

131. If the geocentric hour angle (t) and declination (δ) are given, we have, by substituting (137) in (245),

$$\sin S' = \sin S \, \frac{\sin (\delta' - \gamma).}{\sin (\delta - \gamma)} \qquad (252)$$

for which γ and δ' are to be determined by (134) and (136), or with sufficient accuracy for the present purpose by the formulæ

$$\tan \gamma = \frac{\tan \varphi'}{\cos t}$$

$$\delta - \delta' = \frac{\rho \, \pi \sin \varphi' \sin (\gamma - \delta)}{\sin \gamma}$$

132. *To find the contraction of the vertical semidiameter of the sun or moon produced by atmospheric refraction.*

Since the refraction increases with the zenith distance, the refraction for the centre of the sun or the moon will be greater than that for the upper limb, and that for the lower limb will be greater than that for the centre. The apparent distance of the

limbs is therefore diminished, and the whole disc, instead of being circular, presents an oval figure, the vertical diameter of which is the least, and the horizontal diameter the greatest. The refraction increasing more and more rapidly as the zenith distance increases, the lower half of the disc is somewhat more contracted than the upper half.

The contraction of the vertical semidiameter may be found directly from the refraction table, by taking the difference of the refractions for the centre and the limb.

EXAMPLE.—The true semidiameter of the moon being 16′ 0″, and the apparent zenith distance of the centre 84°, find the contraction of the upper and lower semidiameters in a mean state of the atmosphere (Barom. 30 inches, Therm. 50° F.). We find from Table I.

For apparent zen. dist. of centre,   84° 0′   Refr. = 8′ 28″.0
"   approx.   "   upper limb, 83 44   "   = 8 9 .4
"   "   "   lower   "   84 16   "   = 8 48 .1

Hence,

Approx. contraction upper semid. = 8′ 28″.0 — 8′ 9″.4 = 18″.6
      "   "   lower   "   = 8 48 .1 — 8 28 .0 = 20 .1

These results are but approximate, since we have supposed the apparent zenith distance of the limb to differ from that of the centre by the true semidiameter, whereas they differ only by the apparent or contracted semidiameter. Hence we must repeat as follows:

App. zen. dist. upper limb = 83° 44′ 18″.6   Refr. = 8′ 9″.7
   "   "   lower   "   = 84 15 39 .9   "   = 8 47 .7

Contraction of upper semid. = 8′ 28″.0 — 8′ 9″.7 = 18″.3
     "   lower   "   = 8 47 .7 — 8 28 .0 = 19 .7

Observations at great zenith distances, where this contraction is most sensible, do not usually admit of great precision, on account of the imperfect definition of the limbs and the uncertainty of the refraction itself. It is, therefore, sufficiently exact to assume the contraction of either the upper or lower semidiameter to be equal to the mean of the two. In the above example, which offers an extreme case, if we take the mean

The small triangle PFP' may be regarded as rectilinear and right-angled at F; whence

$$FP' = PP' \times \cos q$$

or

$$\Delta S_q = \Delta S_1 \frac{S_q \cos^2 q}{S_1}$$

If we put S_1 for S_q in the second member, the resulting value of ΔS_q will never be in error $0''.2$ for zenith distances less than $85°$, and it suffices to take

$$\Delta S_q = \Delta S_1 \cos^2 q \qquad (253)$$

This formula is sufficiently exact for all purposes to which we shall have occasion to apply it.

134. *To find the contraction of the horizontal semidiameter.*—The formula (253) for $q = 90°$ makes the contraction of the horizontal semidiameter $= 0$. This results from our having assumed that the portions of vertical circles drawn through the several points of the limb are parallel, and this assumption departs most from the truth in the case of the two vertical circles drawn through the extremities of the horizontal diameter. To investigate the error in this case, let ZM, Fig. 23, be the vertical circle drawn through the centre of the body, ZM' that drawn through the extremity of the horizontal semidiameter MM'. In consequence of the refraction, the points M and M' appear at N and N'. If we denote the zenith distances of M and N by ζ and z, those of M' and N' by ζ' and z', the refraction MN may be expressed as a function either of z or of ζ, Art. 107, and we shall have

Fig. 23.

$$r = k \tan z = k' \tan \zeta$$

where k and k' are given by the refraction table with the arguments z and ζ. The zenith distance of the point M' differs so little from that of M that the values of k and k' will be sensibly the same for both points, and we shall have for the refraction $M'N'$,

$$r' = k \tan z' = k' \tan \zeta'$$

These two equations give

$$\frac{\tan z}{\tan z'} = \frac{\tan \zeta}{\tan \zeta'}$$

But if the triangle ZNN' is right-angled at N, we have

$$\cos Z = \frac{\tan z}{\tan z'}$$

and hence, also,

$$\cos Z = \frac{\tan \zeta}{\tan \zeta'}$$

Therefore the triangle ZMM' is also right-angled, and it gives

$$\tan Z = \frac{\tan S}{\sin (z+r)} = \frac{\tan S'}{\sin z}$$

in which $S = MM'$ and $S' = NN'$.   Hence

$$\frac{\tan S}{\tan S'} = \frac{\sin (z + r)}{\sin z} = \cos r + \sin r \cot z$$

or, very nearly,

$$\frac{S}{S'} = 1 + r \sin 1'' \cot z = 1 + k \sin 1''$$

Hence the contraction of the horizontal semidiameter is expressed by the following formula:

$$S - S' = S' k \sin 1''$$

In the zenith, the mean value of $\log k$ is 1.76156; at the zenith distance 85°, it is 1.71020.   For $S' = 16'$, therefore, the contraction found by this formula is 0''.27 in the zenith, and 0''.24 for 85°.   Thus, *for all zenith distances less than 85° the contraction of the horizontal semidiameter is very nearly constant and equal to one-fourth of a second.*

When the body is in the horizon, we have $k = r \cot z = 0$, and hence $S - S' = 0$, which follows also from the sensible parallelism of the vertical circles at the horizon.

REDUCTION OF OBSERVED ZENITH DISTANCES TO THE CENTRE OF THE EARTH.

135. It is important to observe a proper order in the application of the several corrections which have been treated of in this chapter.

The zenith distance of any point of the heavens observed with any instrument is generally affected with the index error and other instrumental errors. These errors will be treated of in the second volume; here we assume that they have been duly allowed for, and we shall call "observed" zenith distance that which would be obtained with a perfect instrument, and shall denote it by z.

In all cases the first step in the reduction is to find the refraction r $(= a\beta^A\gamma^\lambda \tan z)$ with the argument z, and then $z + r$ is the zenith distance freed from refraction.

1st. In the case of a *fixed star*,

$$\zeta = z + r$$

is at once the required geocentric zen. dist.

2d. In the case of the *moon*, the zenith distance observed is that of the upper or lower limb. If S is the geocentric and S' the augmented semidiameter found by Art. 128, 129, or 130,

$$\zeta' = z + r \pm S'$$

is the apparent zenith distance of the moon's centre freed from refraction, and affected only by parallax, and, consequently, it is that which has been denoted by the same symbol in the discussion of the parallax. With this, therefore, we compute the parallax in zenith distance, $\zeta' - \zeta$, by Art. 95, and then

$$\zeta = \zeta' - (\zeta' - \zeta)$$

is the required geocentric zenith distance of the moon's centre.

To compute S' by (248), (250), or (251), we must first know ζ'; but it will suffice to employ in these formulæ the approximate value $\zeta' = z + r \pm S$.

We can, however, avoid the computation of S', when extreme precision is not required, by computing the parallax for the zenith distance of the limb. Thus, putting $\zeta' = z + r$, and

computing $\zeta' - \zeta$ by Art. 95, the quantity $\zeta = \zeta' - (\zeta' - \zeta)$ is the geocentric zenith distance of the limb; and therefore, applying the geocentric semidiameter, $\zeta \pm S$ is the required geocentric zenith distance of the moon's centre. This process involves the error of assuming the horizontal parallax for the limb to be the same as that for the moon's centre. It can easily be shown, however, that the error in the result will never amount to $0''.2$, which in most cases in practice is unimportant. The exact amount will be investigated in the next article.

3d. In the case of the *sun* or a *planet*, when the limb has been observed, the process of reduction is, theoretically, the same as for the moon; but the parallax is so small that the augmentation of the semidiameter is insensible. We therefore take

$$\zeta' = z + r \pm S$$

and then, computing the parallax by Art. 96, or even by Art. 90, $\zeta = \zeta' - (\zeta' - \zeta)$ is the true geocentric zenith distance.

If a point has been referred to the sea horizon and the measured altitude is H, then, D being the dip of the horizon, $h' = H - D$ is properly the observed altitude, and $z = 90° - h'$ the observed zenith distance, with which we proceed as above.

136. The process above given for reducing the observed zenith distance of the moon's limb to the geocentric zenith distance of the moon's centre, is that which is usually employed; but the whole reduction, exclusive of refraction, may be directly and rigorously computed as follows. Putting

$\zeta' = z + r =$ the apparent zenith distance of the moon's limb corrected for refraction,
$\zeta =$ the geocentric zenith distance of the moon's centre,

then, S' being the augmented semidiameter, we must substitute $\zeta' \pm S'$ for ζ' in the formulæ for parallax, and, by (101), we have

$$f \sin (\zeta' \pm S') = \sin \zeta - \rho \sin \pi \cos (\varphi - \varphi') \tan \gamma$$
$$f \cos (\zeta' \pm S') = \cos \zeta - \rho \sin \pi \cos (\varphi - \varphi')$$

Multiplying the first of these by $\cos \zeta'$, the second by $\sin \zeta'$, and subtracting, we have

be no appreciable error in regarding them as proportional to their sines; and hence we have

$$\zeta' - \zeta = p \mp S \mp \tfrac{1}{2} (p \mp S) \sin p \sin S \qquad (257)$$

the upper signs being used for the upper limb and the lower signs for the lower limb.

In this formula, p is the parallax computed for the zenith distance of the limb, and the small term $\tfrac{1}{2}(p \mp S)\sin p \sin S$ may be regarded as the correction for the error of assuming the parallax of the limb to be the same as that of the centre.

EXAMPLE.—In latitude $\varphi = 38° 59'$ N., given the observed zenith distance of the moon's lower limb, $z = 47° 29' 58''$, the azimuth $A = 33° 0'$, Barom. 30.25 inches, At. Therm. 65° F., Ext. Therm. 64° F., Eq. hor. par. $\pi = 59' 10''.20$; find the geocentric zenith distance of the moon's centre:

(Table III.) $(\varphi - \varphi') = 11' 15''$		
log $(\varphi - \varphi') =$	2.8293	
log cos A	9.9236	
log γ	2.7529	
(Table III.) log ρ	9.999428	
log sin π	8.235806	
log sin $(\zeta' - \gamma)$	9.866652	
log sin p	8.101886	
log sin π	8.235806	
(Art. 128) log (0.272956)	9.436093	
log sin S	7.671899	
log sin p sin S	5.7739	
log $(p + S)$	8.5535	
log $\tfrac{1}{2}$	9.6990	
log $\tfrac{1}{2} (p + S)$ sin p sin S	9.0264	

$z =$	$47° 29' 58''.00$
(Table II.) $r =$	$1\ \ 2 .27$
$\zeta' =$	$47\ \ 31\ \ \ \ 0 .27$
$\gamma =$	$0\ 26 .$
$\zeta' - \gamma =$	$47\ \ 21\ \ 34 .$
$p =$	$43'\ 28''.09$
$S =$	$16\ \ \ 9 .00$
$p + S =$	$59\ 37 .09$
$\tfrac{1}{2} (p + S)$ sin p sin $S =$	$0 .11$
$\zeta' - \zeta =$	$59\ 37 .20$
$\zeta =$	$46° 31' 23''.07$

It is hardly necessary to observe that if the geocentric zenith distance of the centre of the moon or other body is given, the apparent zenith distance of the limb affected by parallax and refraction will be deduced by reversing the order of the steps above explained.

If altitudes are given, we may employ altitudes throughout the computation, putting everywhere $90° - h$, &c. for z, &c., and making the necessary obvious modifications in the formulæ.

CHAPTER V.

FINDING THE TIME BY ASTRONOMICAL OBSERVATIONS.

137. WE have seen, Art. 55, that the local time at any place is readily found when the hour angle of any known heavenly body is given. This hour angle is obtained by observation, but, a direct measure of it being in general impracticable, we must have recourse to observations from which it can be deduced.

The observer is supposed to be provided with a clock, chronometer, or watch, which is required to show the time, mean or sidereal, either at his own or at some assumed meridian, such as that of Greenwich.

The *clock correction** is the quantity which must be added algebraically to the time shown by the clock to obtain the correct time at the meridian for which the clock is regulated. If we put

$$T = \text{the clock time,}$$
$$T' = \text{the true time,}$$
$$\Delta T = \text{the clock correction,}$$

we have

$$T' = T + \Delta T$$

or

$$\Delta T = T' - T \qquad (258)$$

and the clock correction will be *positive* or *negative*, according as the clock is *slow* or *fast*. It is generally the immediate object of an observation for time to determine this correction. At the instant of the observation, the time T is noted by the clock, and if this time agrees with the time T' computed from the observation, the clock is correct; otherwise the clock is in error, and its correction is found by the equation $\Delta T = T' - T$.

The *clock rate* is the daily or hourly increase of the clock correction. Thus, if

---

* For brevity, I shall use *clock* to denote any time-keeper.

$\Delta T_0 =$ the clock correction at a time T_0,

$\Delta T =$ "   "   " T,

$\delta T =$ the clock rate in a unit of time,

we have

$$\Delta T = \Delta T_0 + \delta T\,(T - T_0) \tag{259}$$

where $T - T_0$ must be expressed in days, hours, &c., according as δT is the rate in one day, one hour, &c.

When, therefore, the clock correction and rate have been found at a certain instant T_0, we can deduce the true time from the clock indication T (or "clock face," as it is often called) at any other instant, by the equation

$$T' = T + \Delta T_0 + \delta T\,(T - T_0) \tag{260}$$

If the clock correction has been determined at two different times T_0 and T, the rate is inferred by the equation

$$\delta T = \frac{\Delta T - \Delta T_0}{T - T_0} \tag{261}$$

But these equations are to be used only so long as we can regard the rate as *constant*.

Since such uniformity of rate cannot be assumed for any great length of time, even with the best clocks (although the perform-ance of some of them is really surprising), it is proper to make the interval between the observations for time so small that the rate may be taken as constant for that interval. The length of the interval will depend upon the character of the clock and the degree of accuracy required.

EXAMPLE.—At noon, May 5, the correction of a mean time clock is $- 16^m\ 47^s.30$; at noon, May 12, it is $- 16^m\ 13^s.50$; what is the mean time on May 25, when the clock face is $11^h\ 13^m\ 12^s.6$, supposing the rate to be uniform?

$$
\begin{aligned}
\text{May 5, corr.} &= -16^m\ 47^s.30 \\
\text{" 12, "} &= -16\ \ 13\ .50 \\
\hline
\text{Rate in 7 days} &= \ +\ 33\ .80 \\
\delta T &= \ +\ \ \ 4\ .829
\end{aligned}
$$

Taking, then, as our starting point $T_0 =$ May 12, 0^h, we have

Local sidereal time of transit $= a =$		13^h 17^m	$37^s.92$	
Longitude	$= -$	1 13	56 .	
Greenwich sidereal time	$=$	12 3	41 .92	
March 15, sid. time of mean noon $=$		23 33	5 .87	
Sid. interval from mean noon	$=$	12 30	36 .55	
Reduction to mean time	$=$	$-$ 2	2 .97	
Mean Gr. time of star's local transit $\Big\}$	$= T' =$	12 28	33 .58	
Chronometer time of do.	$= T =$	12 30	19 .00	
Chronometer correction	$= \Delta T = -$	1	45 .42	

140. (B). *Equal altitudes of the sun before and after noon.*—If the declination of the sun were the same at both observations, the hour angles reckoned from the meridian east and west would be equal when the altitudes were equal, and the mean of the two clock times of observation would be the time by the clock at the instant of apparent noon, and we should find the clock correction as in the case of a fixed star. To find the correction for the change of declination, let

$\varphi =$ the latitude of the place of observation,
$\delta =$ the sun's declination at apparent (local) noon,
$\Delta\delta =$ the increase of declination from the meridian to the west observation, or the decrease to the east observation,
$h =$ the sun's true altitude at each observation,
$T_0 =$ the mean of the clock times A.M. and P.M.,
$\Delta T_0 =$ the correction of this mean to reduce to the clock time of apparent noon,
$t =$ half the elapsed time between the observations.

Then we have

$t + \Delta T_0 =$ the hour angle at the A.M. observation reckoned towards the east,
$t - \Delta T_0 =$ the hour angle at the P.M. observation,
$\delta - \Delta\delta \ =$ the declination at the A.M.   "
$\delta + \Delta\delta \ =$ "   "   P.M   "

and, by the first equation of (14) applied to each observation,

$$\sin h = \sin \varphi \sin (\delta - \Delta\delta) + \cos \varphi \cos (\delta - \Delta\delta) \cos (t + \Delta T_0)$$
$$\sin h = \sin \varphi \sin (\delta + \Delta\delta) + \cos \varphi \cos (\delta + \Delta\delta) \cos (t - \Delta T_0)$$

If we substitute

$$\sin (\delta \pm \Delta\delta) = \sin \delta \cos \Delta\delta \pm \cos \delta \sin \Delta\delta$$
$$\cos (\delta \pm \Delta\delta) = \cos \delta \cos \Delta\delta \mp \sin \delta \sin \Delta\delta$$
$$\cos (t \pm \Delta T_0) = \cos t \cos \Delta T_0 \mp \sin t \sin \Delta T_0$$

and then subtract the first equation from the second, we shall find

$$0 = 2 \sin \varphi \cos \delta \sin \Delta\delta - 2 \cos \varphi \sin \delta \sin \Delta\delta \cos t \cos \Delta T_0$$
$$+ 2 \cos \varphi \cos \delta \sin t \cos \Delta\delta \sin \Delta T_0$$

whence, by transposing and dividing by the coefficient of $\sin \Delta T_0$,

$$\sin \Delta T_0 = - \frac{\tan \Delta\delta . \tan \varphi}{\sin t} + \frac{\tan \Delta\delta . \tan \delta}{\tan t} \cos \Delta T_0$$

This is a rigorous expression of the required correction ΔT_0, but the change of declination is so small that we may put $\Delta\delta$ for its tangent, ΔT_0 for its sine, and unity for $\cos \Delta T_0$, without any appreciable error; and, since $\Delta\delta$ is expressed in seconds of arc, we shall obtain ΔT_0 in seconds of time by dividing the second member by 15. We thus find the formula*

$$\Delta T_0 = - \frac{\Delta\delta . \tan \varphi}{15 \sin t} + \frac{\Delta\delta . \tan \delta}{15 \tan t} \qquad (262)$$

The Ephemeris gives the hourly change of δ. If we take it for the Greenwich instant corresponding to the local noon, and call it $\Delta'\delta$, and if t is reduced to hours, we have

$$\Delta\delta = \Delta'\delta . t$$

and our formula becomes

$$\Delta T_0 = - \frac{\Delta'\delta . t \tan \varphi}{15 \sin t} + \frac{\Delta'\delta . t \tan \delta}{15 \tan t} \quad \begin{bmatrix} \text{Equation} \\ \text{for noon.} \end{bmatrix} \quad (263)$$

To facilitate the computation in practice, we put

$$A = - \frac{t}{15 \sin t} \qquad B = \frac{t}{15 \tan t}$$

$$a = A . \Delta'\delta . \tan \varphi \qquad b = B . \Delta'\delta . \tan \delta \qquad \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \quad (264)$$

then we have

$$\Delta T_0 = a + b$$

---

* As first given by GAUSS, *Monatliche Correspondenz*, Vol. 23.

and declination, since these quantities do not enter into the computation. In observations of the sun, an error in the latitude affects the term

$$a = A \Delta' \delta \tan \varphi$$

by differentiating which we find that an error $d\varphi$ produces in a the error $da = A \Delta' \delta . \sec^2 \varphi . d\varphi$, or, putting $\sin d\varphi$ for $d\varphi$,

$$da = A \Delta' \delta \sec^2 \varphi \sin d\varphi$$

In the same manner, we find that an error $d\delta$ in the declination produces in b the error

$$db = B \Delta' \delta \sec^2 \delta \sin d\delta$$

In the example of Art. 140, suppose the latitude and declination were each in error 1'. We have

log $A \Delta' \delta$ n1.2446	log $B \Delta' \delta$ 0.9793
log $\sec^2 \varphi$ 0.2188	log $\sec^2 \delta$ 0.0044
log $\sin$ 1' 6.4637	log $\sin$ 1' 6.4637
log da n7.9271	log db 7.4474
$da = -0^{\circ}.008$	$db = +0^{\circ}.003$

If $d\varphi$ and $d\delta$ had opposite signs, the whole error in this case would be $0^{\circ}.008 + 0^{\circ}.003 = 0^{\circ}.011$. As the observer can always easily obtain his latitude within 1' and the declination (even when the longitude is somewhat uncertain) within a few seconds, we may regard the method as practically free from the effects of any errors in these quantities. The accuracy of the result will therefore depend wholly upon the accuracy of the observations.

The accuracy of the observations depends in a measure upon the constancy of the instrument, but chiefly upon the skill of the observer. Each observer may determine the probable error of his observations by discussing them by the method of least squares. An example of such a discussion will be given in the following article.

The effect of an error in the altitude is given by (266). Since we have, A being the azimuth of the object,

$$\sin A = \frac{\cos \delta \sin t}{\cos h}$$

the formula may also be written

$$\Delta' T_0 = \frac{\Delta h}{30 \cos \varphi \sin A}$$

which will be least when the denominator is greatest, *i.e.* when $A = 90°$ or $270°$, or when the object is near the prime vertical. From this we deduce the practical precept to *take the observations when the object is nearly east or west.* This rule, however, must not be carried so far as to include observations at very low altitudes, where anomalies in the refraction may produce unknown differences in the altitudes. If the star's declination is very nearly equal to the latitude, it will be in the prime vertical only when quite near to the meridian, and then both observations may be obtained within a brief interval of time; and this circumstance is favorable to accuracy, inasmuch as the instrument will be less liable to changes in this short time.

144. *Probable error of observation.*—The error of observation is composed of two errors, one arising from imperfect setting of the index of the sextant, the other from imperfect noting of the time; but these are inseparable, and can only be discussed as a single error in the observed time. The individual observations are also affected by any irregularity of graduation of the sextant, but this error does not affect the mean of a *pair* of observations on opposite sides of the meridian; and therefore the error of observation proper will be shown by comparing the mean of the several pairs with the mean of these means. If, then, the mean of a pair of observed times be called a, the mean of all these means a_0, the probable error of a single pair, supposing all to be of the same weight, is*

$$r = q \sqrt{\frac{\Sigma(a - a_0)^2}{n - 1}}$$

in which $n =$ the number of pairs, and $q = 0.6745$ is the factor to reduce mean to probable errors. The probable error of the final mean a_0 is

$$r_0 = \frac{\epsilon}{\sqrt{n}}$$

---

* See Appendix, *Least Squares.*

EXAMPLE.—At the U. S. Naval Academy, June 18, 1849, the following series of equal altitudes of the sun was observed.

Chro. A.M.	Chro. P.M.	a	$a - a_0$	$(a - a_0)^2$
$0^h 43^m 53^s.$	$9^h 44^m 3^s.5$	$5^h 13^m 58^s.25$	$+ 0^s.12$	0.0144
44 19.	43 38.	58 .50	$+ 0.37$	.1369
44 45.	43 11 .5	58 .25	$+ 0.12$	.0144
45 11.	42 46.3	58 .65	$+ 0.52$	.2704
45 37.	42 19.7	58 .85	$+ 0.22$	.0484
46 1.7	41 53.5	57 .60	$- 0.53$	.2809
46 28 .5	41 27.	57 .75	$- 0.38$	.1444
46 55.	41 0.5	57 .75	$- 0.38$	.1444
47 19.7	40 36.5	58 .10	$- 0.08$	.0009

$$a_0 = 5 \ 13 \ 58 .13 \qquad \Sigma (a - a_0)^2 = 1.0551$$

$$n = 9$$
$$n - 1 = 8$$

$$r = q \sqrt{\frac{\Sigma (a - a_0)^2}{n - 1}} = 0^s.245$$

$$r_0 = \frac{r}{\sqrt{n}} = 0^s.082$$

A similar discussion of a number of sets of equal altitudes of the sun taken by the same observer gave $0^s.23$ as the probable error of a single pair for that observer, and consequently the probable error of the result of six observations on each side of the meridian would be only $0^s.23 \div \sqrt{6} = 0^s.094$. This, however, expresses only the *accidental error of observation*, and does not include the effect of changes in the state of the sextant between the morning and afternoon observations. Such changes are not unfrequently produced by the changes of temperature to which it is exposed in observations of the sun; it is important, therefore, to guard the instrument from the sun's rays as much as possible, and to expose it only during the few minutes required for each observation. The determination of the time by stars is mostly free from difficulties of this kind, but the observation is not otherwise so accurate as that of the sun, except in the hands of very skilful observers.

THIRD METHOD.—BY A SINGLE ALTITUDE, OR ZENITH DISTANCE.

145. Let the altitude of any celestial body be observed with the sextant or any altitude instrument, and the time noted by the clock. For greater precision, observe several altitudes in quick succession, noting the time of each, and take the mean of the altitudes as corresponding to the mean of the times. But

in taking the mean of several observations in this way, it must not be forgotten that we assume that the altitude varies in proportion to the time, which is theoretically true only in the exceptional case where the observer is on the equator and the star's declination is zero. It is, however, practically true for an interval of a few minutes when the star is not too near the meridian. The observations themselves will generally show the limit beyond which it will not be safe to apply this rule. When the observations have been extended beyond this limit, a correction for the unequal change in altitude (*i.e.* for second differences) can be applied, which will be treated of below.

With the altitude and azimuth instrument we generally obtain zenith distances directly. In all cases, however, we may suppose the observation to give the zenith distance. Having then corrected the observation for instrumental errors, for refraction, &c., Arts. 135, 136, let ζ be the resulting true or geocentric zenith distance. Let φ be the latitude of the place of observation, δ the star's declination, t the star's hour angle. The three sides of the spherical triangle formed by the zenith, the pole, and the star may be denoted by $a = 90° - \varphi$, $b = \zeta$, $c = 90° - \delta$, and the angle at the pole by $B = t$, and hence, Art. 22, we deduce

$$\sin \tfrac{1}{2} t = \sqrt{\left(\frac{\sin \tfrac{1}{2}[\zeta + (\varphi - \delta)] \sin \tfrac{1}{2}[\zeta - (\varphi - \delta)]}{\cos \varphi \cos \delta} \right)} \qquad (267)$$

which gives t by a very simple logarithmic computation. From t we deduce, by Art. 55, the local time, which compared with the observed clock time gives the clock correction required.

It is to be observed that the double sign belonging to the radical in (267) gives two values of $\sin \tfrac{1}{2} t$, the positive corresponding to a west and the negative to an east hour angle; since any given zenith distance may be observed on either side of the meridian. To distinguish the true solution, the observer must of course note on which side of the meridian he has observed.

If the object observed is the sun, the moon, or a planet, its declination is to be taken from the Ephemeris, for the time of the observation (referred to the meridian of the Ephemeris); but, as this time is itself to be found from the observation, we must at first assume an approximate value of it, with which an approximate declination is found. With this declination a first compu-

$$
\begin{array}{rl}
\text{Observed } 2\odot = & 33°\ 10'\ \ 0'' \\
\text{Index corr.} \ = & -\ \ \ 1\ 10 \\
\hline
& 33\ \ \ 8\ 50 \\
\text{App. altitude} \ = & 16\ 34\ 25 \\
\hline
z = & 73\ 25\ 35 \\
\text{(Table II.)}\, r = & +\ \ \ 3\ 15 \\
\pi \sin z = p = & -\ \ \ \ \ \ \ 8 \\
S = & -\ 16\ 17 \\
\hline
\zeta = & 73\ \ 12\ 25
\end{array}
$$

The computation by (267) is then as follows:

$$
\begin{array}{rll}
\varphi = & 38°\ 58'\ 53'' \\
\delta = & -\ 22\ 50\ 27 \\
\hline
\varphi - \delta = & 61\ 49\ 20 \\
\zeta = & 73\ 12\ 25 \\
\hline
\tfrac{1}{2}\text{ sum} = & 67\ 30\ 52\ .5 \\
\tfrac{1}{2}\text{ diff.} = & \ 5\ 41\ 32\ .5
\end{array}
\qquad
\begin{array}{rl}
\log \sec \varphi & 0.109383 \\
\log \sec \delta & 0.035464 \\
\log \sin \tfrac{1}{2}\text{ sum} & 9.965661 \\
\log \sin \tfrac{1}{2}\text{ diff.} & 8.996455 \\
\hline
& 19.106963 \\
\log \sin \tfrac{1}{2} t & 9.553482
\end{array}
$$

$$
\begin{array}{rl}
\tfrac{1}{2} t = & 20°\ 57'\ 25''.6 \\
\text{Apparent time} = t = & 2^{h}\ 47^{m}\ 39^{s}.4 \\
\text{Eq. of time} \ \ = & -\ \ 7\ \ 25\ .8 \\
\text{Local mean time} \ = & 2\ \ 40\ \ 13\ .6 \\
\text{Longitude} \ = & 5\ \ \ 5\ \ 57\ .5 \\
\hline
\text{True Gr. Time} = T' = & 7\ \ 46\ \ 11\ .1 \\
T = & 7\ \ 36\ \ 35\ .1 \\
\hline
\Delta T = & +\ \ 9\ \ 36\ .0
\end{array}
$$

agreeing so nearly with the assumed correction that a repetition of the computation is unnecessary.

146. If it is preferred to use the altitude instead of the zenith distance, put the true altitude $h = 90° - \zeta$, and the polar distance of the star $P = 90° - \delta$, then we have, in (267),

$$
\sin \tfrac{1}{2}[\zeta - (\varphi - \delta)] = \sin \tfrac{1}{2}(90° - h - \varphi + 90° - P) = \cos \tfrac{1}{2}(h + \varphi + P)
$$
$$
\sin \tfrac{1}{2}[\zeta + \varphi - \delta \,] = \sin \tfrac{1}{2}(90° - h + \varphi - 90° + P) = \sin \tfrac{1}{2}(\varphi + P - h)
$$

If then we put

$$
s = \tfrac{1}{2}(h + \varphi + P)
$$

the formula becomes

$$
\sin \tfrac{1}{2} t = \sqrt{\left(\frac{\cos s \, \sin (s - h)}{\cos \varphi \, \sin P} \right)} \qquad (268)
$$

for the logarithms of sin φ sin δ and cos φ cos δ will be constant, and for each observation we shall only have to take from the trigonometric table the log. of cos ζ; the logarithm of the numerator will then be found by the aid of ZECH's Addition or Subtraction Table, which is included in HULSSE's edition of VEGA's Tables. The addition or the subtraction table will be used according as sin φ sin δ is positive or negative.

149. *Effect of errors in the data upon the time computed from an altitude.*—We have from the differential equation (51), Art. 35, multiplying dt by 15 to reduce it to seconds of arc,

$$\sin q \cos \delta\ (15\,dt) = d\zeta - \cos A\,d\varphi + \cos q\,d\delta$$

where $d\zeta$, $d\varphi$, $d\delta$, may denote small errors of ζ, φ, δ, and dt the corresponding error of t; A is the star's azimuth, q the parallactic angle, or angle at the star.

If the zenith distance alone is erroneous, we have, by putting $d\varphi = 0$, and $d\delta = 0$,

$$15\,dt = \frac{d\zeta}{\sin q \cos \delta} = \frac{d\zeta}{\cos \varphi \sin A}$$

from which it follows that a given error in the zenith distance will have the least effect upon the computed time when the azimuth is 90° or 270°; that is, when the star is on the prime vertical; for we then have $\sin A = \pm 1$, and the denominator of this expression obtains its maximum numerical value. Also, since cos φ is a maximum for $\varphi = 0$, it follows that observations of zenith distances for determining the time give the most accurate results when the place is on the equator. On the other hand, the least favorable position of the star is when it is on the meridian, and the least favorable position of the observer is at the pole.

By putting $d\zeta = 0$, $d\delta = 0$, $\sin q \cos \delta = \cos \varphi \sin A$ we have

$$15\,dt = -\frac{d\varphi}{\cos \varphi \tan A}$$

by which we see that an error in the latitude also produces the least effect when the star is on the prime vertical, or when the observer is on the equator. Indeed, when the star is exactly in

the prime vertical, a small error in φ has *no* appreciable effect : since, then, tan $A = \infty$, and hence when the latitude is uncertain, we may still obtain good results by observing only stars near the prime vertical.

By putting $d\zeta = 0$, $d\varphi = 0$, we have

$$15\,dt = \frac{d\delta}{\cos \delta \tan q}$$

which shows that the error in the declination of a given star produces the least effect when the star is on the prime vertical;* and of different stars the most eligible is that which is nearest to the equator.

As very great zenith distances (greater than 80°) are, if possible, to be avoided on account of the uncertainty in the refraction, the observer will often be obliged, especially in high latitudes, to take his observations at some distance from the prime vertical, in which case small errors of zenith distance, latitude, or declination may have an important effect upon the computed *clock correction*. Nevertheless, *constant* errors in these quantities will have no sensible effect upon the *rate* of the clock deduced from zenith distances of the same star on different days, if the star is observed at the same or nearly the same azimuth, on the same side of the meridian; for all the clock corrections will be increased or diminished by the same quantities, so that their differences, and consequently the rate, will be the same as if these errors did not exist. The errors of eccentricity and graduation of the instrument are among the constant errors which may thus be eliminated.

But if the same star is observed both east and west of the meridian, and at the same distance from it, sin A or tan A, and tan q, will be positive at one observation and negative at the other, and, having the same numerical value, constant errors $d\varphi$, $d\delta$, and $d\zeta$ will give the same numerical value of dt with opposite signs. Hence, while one of the deduced clock corrections will be too great, the other will be too small, and their mean will be the true correction at the time of the star's transit

* From the equation sin $q = \dfrac{\cos \delta}{\cos \delta} \sin A$, it follows that sin q is a maximum (for constant values of φ and δ) when sin $A = 1$, and tan q is a maximum in the same case.

over the meridian.   Hence, it follows again, as in Art. 143, that small errors in the latitude and declination have no sensible effect upon the time computed from equal altitudes.

150. *To find the change of zenith distance of a star in a given interval of time, having regard to second differences.*

The formula

$$d\zeta = \cos \varphi \sin A \, dt$$

is strictly true only when $d\zeta$ and dt are infinitesimals.   But the complete expression of the finite difference $\Delta\zeta$ in terms of the finite difference Δt involves the square and higher powers of Δt. Let ζ be expressed as a function of t of the form

$$\zeta = ft$$

then, to find any zenith distance $\zeta + \Delta\zeta$ corresponding to the hour angle $t + \Delta t$, we have, by TAYLOR's Theorem,

$$\zeta + \Delta\zeta = f(t + \Delta t) = ft + \frac{dft}{dt} \cdot \Delta t + \frac{d^2ft}{dt^2} \cdot \frac{\Delta t^2}{2} + \cdots$$

or, taking only second differences,

$$\Delta\zeta = \frac{d\zeta}{dt} \cdot \Delta t + \frac{d^2\zeta}{dt^2} \cdot \frac{\Delta t^2}{2}$$

We have already found

$$\frac{d\zeta}{dt} = \cos \varphi \sin A$$

which gives, since A varies with t, but φ is constant,

$$\frac{d^2\zeta}{dt^2} = \cos \varphi \cos A \cdot \frac{dA}{dt}$$

But from the second of equations (51) we have, since $d\delta$ and $d\varphi$ are here zero,

$$\frac{dA}{dt} = \frac{\cos q \cos \delta}{\sin \zeta} = \frac{\cos q \sin A}{\sin t}$$

whence

$$\frac{d^2\zeta}{dt^2} = \frac{\cos \varphi \sin A \cos A \cos q}{\sin t}$$

and the expression for $\Delta\zeta$ becomes

$$\Delta\zeta = \cos\varphi\,\sin A \cdot \Delta t + \frac{\cos\varphi\,\sin A\,\cos A\,\cos q}{\sin t} \cdot \frac{\Delta t^2}{2}$$

Since $\Delta\zeta$ and Δt are here supposed to be expressed in parts of the radius, if we wish to express them in seconds of arc and of time respectively, we must substitute for them $\Delta\zeta\sin 1''$ and $15\Delta t\sin 1''$, and the formula becomes

$$\Delta\zeta = \cos\varphi\,\sin A\,(15\Delta t) + \frac{\cos\varphi\,\sin A\,\cos A\,\cos q}{\sin t} \cdot \frac{(15\Delta t)^2\sin 1''}{2} \qquad (271)$$

But in so small a term as the last we may put

$$\frac{(15\Delta t)^2\sin 1''}{2} = \frac{2\sin^2\tfrac{1}{2}\Delta t}{\sin 1''} = m$$

the value of which is given in our Table V., and its logarithm in Table VI.; so that if we put also

$$a = \cos\varphi\,\sin A, \qquad k = \frac{\cos A\,\cos q}{\sin t}$$

we shall have

$$\Delta\zeta = 15\,a\Delta t + akm \qquad (272)$$

151. *A number of zenith distances being observed at given clock times, to correct the mean of the zenith distances or of the clock times for second differences.*—The first term of the above value of $\Delta\zeta$ varies in proportion to Δt, but the second term varies in proportion to Δt^2; and hence, when the interval is sufficiently great to render this second term sensible, equal intervals of time correspond to unequal differences of zenith distance, and *vice versa*: in other words, we shall have second differences either of the zenith distance or of the time. Two methods of correction present themselves.

1st. *Reduction of the mean of the zenith distances to the mean of the times.*—Let T_1, T_2, T_3, &c. be the observed clock times; ζ_1, ζ_2, ζ_3, &c. the corresponding observed zenith distances; T the mean of the times; ζ_0 the mean of the zenith distances; ζ the zenith distance corresponding to T. The change $\zeta_1 - \zeta$ corresponds to the interval $T_1 - T$, $\zeta_2 - \zeta$ to $T_2 - T$, &c.; so that if we put

$$T_1 - T = \tau_1, \quad T_2 - T = \tau_2, \text{&c.}$$

we have, by (272),

$$\zeta_1 - \zeta = 15 \, a \, \tau_1 + akm_1$$
$$\zeta_2 - \zeta = 15 \, a \, \tau_2 + akm_2$$
$$\zeta_3 - \zeta = 15 \, a \, \tau_3 + akm_3$$
$$\&\text{c.} \qquad \&\text{c.}$$

in which $m_1 = \dfrac{2 \sin^2 \frac{1}{2} \tau_1}{\sin 1''}$, $m_2 = \dfrac{2 \sin^2 \frac{1}{2} \tau_2}{\sin 1''}$, &c., are found by Tab. V.

with the arguments τ_1, τ_2, &c. The mean of these equations, observing that

$$\tau_1 + \tau_2 + \tau_3 + \&\text{c.} = 0$$

gives

$$\zeta = \zeta_0 - ak \cdot \frac{m_1 + m_2 + m_3 + \&\text{c}}{n}$$

in which $n =$ the number of observations. Or, denoting the mean of the values of m from the table by m_0, that is, putting

$$m_0 = \frac{m_1 + m_2 + m_3 + \&\text{c.}}{n}$$

we have

$$\zeta = \zeta_0 - akm_0 \qquad (273)$$

2d. Reduction of the mean of the times to the mean of the zenith distances.—Let T_0 be the clock time corresponding to the mean of the zenith distances, then $\zeta_0 - \zeta$ is the change of zenith distance in the interval $T_0 - T$, and, since this interval is very small, we shall have sensibly

$$15 \, a \, (T_0 - T) = \zeta_0 - \zeta = akm_0$$

whence

$$T_0 = T + \tfrac{1}{15} km_0 \qquad (274)$$

We have, then, only to compute the true time T_0' from the mean of the zenith distances in the usual manner, and the clock correction will then be found, as in other cases, by the formula

$$\Delta T = T_0' - T_0$$

To compute k, we must either first find q and A, or, which is preferable, express it by the known quantities. We have

$$\cos q \cos A = \cos t - \sin q \sin A \cos \zeta$$
$$= \cos t - \frac{\sin^2 t}{\sin^2 \zeta} \cos \varphi \cos \delta \cos \zeta$$

whence

$$T_0 = T + \tfrac{1}{15} m_0 \cot t - \tfrac{1}{15} m_0 \frac{\sin t \cos \varphi \cos \delta}{\sin \zeta \tan \zeta} \qquad (275)$$

in which we employ for ζ and t the mean zenith distance and the computed hour angle.

This mode of correction is evidently more simple and direct than the first.

EXAMPLE.—In St. Louis, Lat. 38° 38′ 15″ N., Long. 6ʰ 1ᵐ 7ˢ W., tne following double altitudes of the sun were observed with a Pistor and Martin prismatic sextant, the index correction of which was + 20″. The assumed correction of the chronometer to mean local time was + 2ᵐ 12ˢ. Barom. 30.25 inches, Att. Therm. 80°, Ext. Therm. 81°.

St. Louis, June 24, 1861.

	2☉		Chronom.		τ		m
	125° 15′ 10″.		22ʰ 14ᵐ 30ˢ.5		6ᵐ 42ˢ		88″.14
	125 49 10		16 7.5		5 5		50 .73
	126 23 0		17 46.0		3 26		23 .14
	126 41 40		18 39.5		2 33		12 .76
	127 32 30		21 6.5		0 6		0 .02
	127 57 45		22 22.		1 10		2 .67
	128 22 0		23 33.5		2 21		10 .84
	128 51 50		25 1.2		3 49		28 .60
	129 8 35		25 51.3		4 39		42 .45
	129 33 0		27 8.5		5 51		67 .19
Mean	127 33 28	$T =$	22 21 12.15			$m_0 =$	82 .65
	+ 20	Correction for }					
	127 33 48	second diff. } $= -$	1.67		log m_0		1.5139
Obs'd ☉	63 46 54	$T_0 =$	22 21 10.48		log. $\tfrac{1}{15}$		8.8239
(*) $\bar{r} =$	− 27 .2	$T_0' =$	22 23 22.78		log cot t		n0.3368
$p =$	+ 8 .7	$\Delta T =$	+ 2 12.30		− 4ˢ.73		n0.6746
$S =$	+ 15′ 46 .8						
$h_0 =$	64 2 16 .8				log $\tfrac{1}{15} m_0$		0.3378
$\zeta_0 =$	25 57 43 .2				log sin t		n9.6215
$\varphi =$	38 38 15 .				log cos ϕ		9.8927
$\delta =$	23 23 49 .3				log cos δ		9.9627
$t =$	− 24° 43′ 48″.4				log cosec ζ_0		0.3588
	$= −$ 1ʰ 38ᵐ 55ˢ.23				log cot ζ_0		0.3378
App. time =	22 21 4.77				− 3ˢ.06		n0.4859
Eq. of time =	+ 2 18.01				− 1.67		
$T_0' =$	22 23 22.78						

* The refraction should here be the mean of the refractions computed for the

The correction for second differences is particularly useful in reducing series of altitudes observed with the repeating circle;* for with this instrument we do not obtain the several altitudes, but only their mean. (See Vol. II.) When the several altitudes are known, we can avoid the correction by computing each observation, or by dividing the whole series into groups of such extent that within the limits of each the second differences will be insensible, and computing the time from the mean of each group.

FOURTH METHOD.—BY THE DISAPPEARANCE OF A STAR BEHIND A TERRESTRIAL OBJECT.

152. The *rate* of the clock may be found by this method with considerable accuracy without the aid of astronomical instruments. The terrestrial object should have a sharply defined vertical edge, behind which the disappearance is to be observed, and the position of the eye of the observer should be precisely the same at all the observations. If the star's right ascension and declination are constant, the difference between the sidereal clock times T_1 and T_2 of two disappearances is the rate δT in the interval, or

$$\delta T = T_1 - T_2$$

but if the right ascension α has increased in the interval by $\Delta\alpha$, then the rate is

$$\delta T = T_1 - T_2 + \Delta\alpha$$

To find the correction for a small change of declination $= \Delta\delta$,

---

several altitudes or zenith distances, but for small zenith distances the difference will be insensible. At great zenith distances we should compute the several refractions, but under 80° we may take the refraction r for the mean apparent zenith distance z_0, and correct it as follows: Take the difference between z_0 and each z, and the mean m_0 of the values of

$$m = \frac{2 \sin^2 \frac{1}{2} (z - z_0)}{\sin 1''}$$

from Table V. (converting the argument $z - z_0$ into time); then the mean of the refractions will be found by the formula

$$r_0 = r + 2m_0 \sin r \sec^2 z_0.$$

The difference $z - z_0$ should not much exceed 1°.

* This method was frequently practised in the geodetic survey of France. See *Nouvelle Description Géométrique de la France* (PUISSANT), Vol. I. p. 96.

we have, by the second equation of (51), since the azimuth is here constant as well as the latitude, so that $dA = 0$ and $d\varphi = 0$,

$$\Delta t = -\frac{\Delta \delta \tan q}{15 \cos \delta}$$

and hence the rate in the interval will be

$$\delta T = T_1 - T_s + \Delta a - \frac{\Delta \delta \tan q}{15 \cos \delta} \tag{276}$$

The angle q will be found with sufficient precision from an approximate value of t by (19) or (20).

If we know the absolute azimuth of the object, we can find the hour angle by Art. 12, and hence also the clock correction.

TIME OF RISING AND SETTING OF THE STARS.

153. *To find the time of true rising or setting,*—that is, the instant when the star is in the true horizon,—we have only to compute the hour angle by the formula (28)

$$\cos t = -\tan \varphi \tan \delta$$

and then deduce the local time by Art. 55.

154. *To find the time of apparent rising or setting,*—that is, the instant when the star appears on the horizon of the observer,—we must allow for the horizontal refraction. Denoting this refraction by r_0, the true zenith distance of the star at the time of apparent rising or setting is $90° + r_0$, and, employing this value for ζ, we compute the hour angle by (267).

Since the altitude $h = 90° - \zeta$, we have in this case $h = -r_0$, with which we can compute the hour angle by the formula (268).

In common life, by the time of sunrise or sunset is meant the instant when the sun's upper limb appears in the horizon. The true zenith distance of the centre is, then, $\zeta = 90° + r_0 - \pi + S$ (where $\pi =$ the horizontal parallax and $S =$ the semidiameter), with which we compute the hour angle as before. The same form is to be used for the moon.

TIME OF THE BEGINNING AND ENDING OF TWILIGHT.

155. Twilight begins in the morning or ends in the evening when the sun is 18° below the horizon, and consequently the

zenith distance is then $\zeta = 90° + 18°$, or $h = -18°$, with which we can find the hour angle by (267) or (268).

Note.—Methods of finding at once both the time and the latitude from observed altitudes will be treated of under Latitude, in the next chapter.

FINDING THE TIME AT SEA.

First Method.—By a Single Altitude.

156. This is the most common method among navigators, as altitudes from the sea horizon are observed with the greatest facility with the sextant. Denoting the observed altitude corrected for the index error of the sextant by H, the dip of the horizon by D, we have the apparent altitude $h' = H - D$; then, taking the refraction r for the argument h', the true altitude of a star is $h = h' - r$. A planet is observed by bringing the estimated centre of its reflected image upon the horizon, so that no correction for the semidiameter is employed; the parallax is computed by the simple formula (π being the horizontal parallax)

$$p = \pi \cos h'$$

and hence for a planet

$$h = h' - r + \pi \cos h'$$

The moon and sun are observed by bringing the reflected image of either the upper or the lower limb to touch the horizon. As very great precision is neither possible nor necessary in these observations, the compression of the earth is neglected, and the parallax is computed by the formula

$$p = \pi \cos (h' - r)$$

and then, S being the semidiameter,

$$h = h' - r + \pi \cos (h' - r) \pm S$$

In nautical works, the whole correction of the moon's altitude for parallax and refraction $= \pi \cos (h' - r) - r$ is given in a table with the arguments apparent altitude (h') and horizontal parallax (π). In the construction of this table the mean refraction is used, but the corrections for the barometer and thermometer are given in a very simple table, although they are not usually of sufficient importance to be regarded in correcting altitudes of the moon which are taken to determine the local time.

ship had not changed her latitude or had run upon a parallel. Hence $\frac{1}{2}\Delta t$ is to be subtracted from the mean of the chronometer times to obtain the chronometer time of the star's transit over the middle meridian.

In this formula we must observe the sign of tan A. It will be more convenient in practice to disregard the signs, and to apply the numerical value of the correction to the middle time according to the following simple rule:—add the correction when the ship has *receded* from the sun; subtract it when the ship has *approached* the sun.

The azimuth may be found by the formula

$$\sin A = \frac{\sin t \cos \delta}{\cos h}$$

in which for t we take one-half the elapsed time.

The sun being the only object which is employed in this way, we should also apply the equation of equal altitudes, Art. 140; but, as the greatest change of the sun's declination in one hour is about $1'$, and the change of the ship's latitude is generally much greater, the equation is commonly neglected as relatively unimportant in a method which at sea is necessarily but approximate. But, if required, the equation may be computed and applied precisely as if the ship had been at rest.

EXAMPLE.—At sea, March 20, 1856, the latitude at noon being $39°$ N., the same altitude was observed A.M. and P.M. as follows, by a chronometer regulated to mean Greenwich time:

Obsd. ☉	30° 0′	A.M. Chro. time	=	11ʰ 39ᵐ 33
Index corr.	— 2	P.M. " "	=	6 20 17
Dip	— 4	Elapsed time $= 2t =$		6 40 44
Refraction	— 2	Middle time	=	2 59 55
Semidiam.	+ 16	Chron. correction	=	— 2 12
	$h = 30$ 8	Green. time of noon	=	2 57 43

The ship changed her latitude between the two observations by $\Delta\varphi = -20' = -1200''$. For the Greenwich date March 20, 2ʰ 58ᵐ, the Ephemeris gives $\delta = +0° 4'$, and we have $t = 3ʰ 20ᵐ 22ˢ = 50° 5' 30''$, $\varphi = 39° 0'$. Hence

log sin t	9.8848	log $\frac{1}{30}$	8.5229
log cos δ	0.0000	log $\Delta\varphi$	3.0792
log sec h	0.0631	log sec φ	0.1095
log sin A	9.9479	log cot A	9.7165
		log 26ˢ.8	1.4281

The ship has approached the sun, and hence 26ˢ.8 must be subtracted from the middle time.

If we wish to apply the equation of equal altitudes, we have further from the Ephemeris $\Delta'\delta = + 59''$, and hence, by Art. 140,

log A	n9.4628	log B	9.2698
log $\Delta'\delta$	1.7709	log $\Delta'\delta$	1.7709
log tan φ	9.9084	log tan δ	7.0658
$a = -$ 13ˢ.9 log a	n1.1421	$b = +$ 0ˢ.0 log b	8.1065

Hence we have

Chro. middle time	$=$	2ʰ 59ᵐ 55ˢ.
Corr. for change of lat. $=$		$-$ 26 .8
Equation of eq. alts.	$=$	$-$ 13 .9
Chro. time app. noon	$=$	2 59 14 .3

At sea, instead of using the observation to find the chronometer correction, we use it to determine the ship's longitude (as will be fully shown hereafter); and therefore, to carry the operation out to the end, we shall have

Chro. time app. noon	$=$	2ʰ 59ᵐ 14ˢ.
Corr. of chronom.	$= -$	2 12 .
Green. mean time noon	$=$	2 57 2 .3
Equation of time	$= -$	7 48 .
Greenwich app. time at the local noon $=$		2 49 14

which is the longitude of the middle meridian, or the longitude of the ship at noon.

159. In low latitudes (as within the tropics) observations for the time may be taken when the sun is very near the meridian, for the condition that the sun should be near the prime vertical may then be satisfied within a few minutes of noon; and in case the ship's latitude is *exactly* equal to the declination, it will be satisfied *only* when the sun is on the meridian in the zenith. In such cases the two equal altitudes may be observed within a few minutes of each other, and all corrections, whether for change of latitude or change of declination, may be disregarded.

CHAPTER VI.

FINDING THE LATITUDE BY ASTRONOMICAL OBSERVATIONS.

160. By the definition, Art. 7, the latitude of a place on the surface of the earth is the *declination of the zenith*. It was also shown in Art. 8 to be equal to the *altitude of the north pole* above the horizon of the place. In adopting the latter definition, it is to be remembered that a depression below the horizon is a negative altitude, and that south latitude is negative. The south latitude of a place, considered numerically, or without regard to its algebraic sign, is equal to the elevation of the south pole.

It is to be remembered, also, that the latitude thus defined is not an angle at the centre of the earth measured by an arc of the meridian, as it would be if the earth were a sphere; but it is the angle which the vertical line at the place makes with the plane of the equator, Art. 81.

We have seen, Art. 86, that there are abnormal deviations of the plumb line, which make it necessary to distinguish between the *geodetic* and the *astronomical* latitude. We shall here treat exclusively of the methods of determining the astronomical latitude; for this depends only upon the actual position of the plumb line, and is merely the declination of that point of the heavens towards which the plumb line is directed.

FIRST METHOD.—BY MERIDIAN ALTITUDES OR ZENITH DISTANCES.

161. Let the altitude or zenith distance of a star of known declination be observed at the instant when it is on the meridian. Deduce the true geocentric zenith distance ζ, and let δ be the geocentric declination, φ the astronomical latitude.

Let the celestial sphere be projected on the plane of the meridian, and let ZNZ', Fig. 24, be the celestial meridian; C the centre of the sphere coincident with that of the earth; PCP' the axis of the sphere; P the north pole; and ECQ the projection

directly the zenith distance, or its supplement, the nadir distance. With a meridian circle *perfectly adjusted in the meridian,* the instant of transit would be known without reference to the clock, and the observation would be made at the instant the star passed the middle thread of the reticule; but when the instrument is not exactly in the meridian, or when the observation is not made on the middle thread, the observed zenith distance must be reduced to the meridian, for which see Vol. II., Meridian Circle.

With the sextant or other portable instruments the meridian altitude of a fixed star may be distinguished as the greatest altitude, and no reference to the time is necessary. But, as the sun, moon, and planets constantly change their declination, their greatest altitudes may be reached either before or after the meridian passage;* and in order to observe a strictly meridian altitude the clock time of transit must be previously computed and the altitude observed at that time.

EXAMPLE 1.—On March 1, 1856, in Long. $10^h\ 5^m\ 32^s$ E., suppose the apparent meridian altitude of the sun's upper limb, north of the zenith, is 63° 49′ 50″, Barom. 30. in., Ext. Therm. 50°; what is the latitude?

$$
\begin{array}{rl}
\text{App. zen. dist. } \odot = & \ \ 26°\ 10′\ 10″. \\
r = & +\ \ \ 28\ .7 \\
p = & -\ \ \ \ \ 3\ .8 \\
S = & +\ 16\ \ 10\ .3 \\
\hline
\zeta = & -\ 26\ \ 26\ \ 45\ .2 \\
\delta = & -\ \ 7\ \ 33\ \ \ 5\ .8 \\
\hline
\varphi = & -\ 33\ \ 59\ \ 51\ .0
\end{array}
$$

EXAMPLE 2.—July 20, 1856, suppose that at a certain place the true zenith distances of α *Aquilæ* south of the zenith, and α *Cephei* north of the zenith, have been obtained as follows:

α *Aquilæ*	α *Cephei*
$\zeta = +\ 26°\ 34′\ 27″.5$	$\zeta = -\ 26°\ 54′\ 28″.3$
$\delta = +\ \ \ 8\ \ \ 29\ \ 22\ .7$	$\delta = +\ 61\ \ 58\ \ 21\ .1$
$\varphi = +\ 35\ \ \ \ \ 3\ \ 50\ .2$	$\varphi = +\ 35\ \ \ \ \ 3\ \ 52\ .8$

The mean latitude obtained by the two stars is, therefore, $\varphi = +\ 35°\ 3′\ 51″.5$. In this example, the stars being at nearly

---

* See Art. 172 for the method of finding the time of the sun's greatest altitude, which may also be used for the moon or a planet.

the same zenith distance, but on opposite sides of the zenith, any constant though unknown error of the instrument, peculiar to that zenith distance, is eliminated in taking the mean. Thus, if the zenith distance in both cases had been $10''$ greater, we should have found from α *Aquilæ* $\varphi = 35°\ 4'\ 0''.2$, but from α *Cephei* $\varphi = 35°\ 3'\ 42''.8$, but the mean would still be $\varphi = 35°\ 3'\ 51''.5$.

It is evident, also, that errors in the refraction, whether due to the tables or to constant errors of the barometer and thermometer, or to any peculiar state of the air common to the two observations, are nearly or quite eliminated by thus combining a pair of stars the mean of whose declinations is nearly equal to the declination of the zenith. The advantages of such a combination do not end here. If we select the two stars so that the difference of their zenith distances is so small that it may be measured with a micrometer attached to a telescope which is so mounted that it may be successively directed upon the two stars without disturbing the angle which it makes with the vertical line, we can dispense altogether with a graduated circle, or, at least, the result obtained will be altogether independent of its indications. For, let ζ and ζ' be the zenith distances, δ and δ' the declinations of the two stars, the second of which is north of the zenith; then, if ζ' denotes only the numerical value of the zenith distance, we have

$$\varphi = \delta + \zeta$$
$$\varphi = \delta - \zeta'$$

the mean of which is

$$\varphi = \tfrac{1}{2}(\delta + \delta') + \tfrac{1}{2}(\zeta - \zeta') \qquad (279)$$

so that the result depends only upon the given declinations and the observed *difference* of zenith distance which is measured with the micrometer. Such is the simple principle of the method first introduced by Captain TALCOTT, and now extensively used in this country. To give it full effect, the instrument formerly known as the *Zenith Telescope* in England has received several important modifications from our Coast Survey. It will be fully treated of, in its present improved form, in Vol. II., where also will be found a discussion of TALCOTT's method in all its details.

162. *Meridian altitudes of a circumpolar star observed both above and below the pole.*—Every star whose distance from the elevated

pole is less than the latitude may be observed at both its upper and lower culminations. If we put

h = the true altitude at the upper culmination,
h_1 =   "   "   "   lower   "
p = the star's polar distance at the upper culmination,
p_1 =   "   "   "   "   lower   "

we have, evidently,

$$\varphi = h - p$$
$$\varphi = h_1 + p_1$$

the mean of which is

$$\varphi = \tfrac{1}{2}(h + h_1) + \tfrac{1}{2}(p_1 - p) \qquad (280)$$

whence it appears that by this method the absolute values of p and p_1 are not required, but only their difference $p_1 - p$. The change of a star's declination by precession and nutation is so small in 12^h as usually to be neglected, but for extreme precision ought to be allowed for. This method, then, is free from any error in the declination of the star, and is, therefore, employed in all fixed observatories.

EXAMPLE.—With the meridian circle of the Naval Academy the upper and lower transits of *Polaris* were observed in 1853 Sept. 15 and 16, and the altitudes deduced were as below:

	Upper Transit.		Lower Transit.	
Sept. 15, App. alt.	40° 28′ 25″.42	Sept. 16,		37° 31′ 39″.76
Barom. 30.005		Barom. 30.146		
Att. Therm. 65°.2 } Ref.	1 6 .27	Att. Therm. 75° } Ref.		1 12 .22
Ext. " 63 .8		Ext. " 74 .6		
	$h = $ 40 27 19 .15		$h_1 = $ 37 30 27 .54	
	$p = $ 1 28 26 .04		$p_1 = $ 1 28 25 .86	
	$\phi = $ 38 58 53 .11		$\phi = $ 38 58 53 .40	
			" " 53 .11	
		Mean $\phi = $ 38 58 53 .26		

In order to compare the results, each observation is carried out separately. By (280) we should have

$$\tfrac{1}{2}(h + h_1) = 38° 58′ 53″.35$$
$$\tfrac{1}{2}(p_1 - p) = \qquad\quad - \ 0 \ .09$$
$$\varphi = 38 \quad 58 \quad 53 \ .26$$

This method is still subject to the whole error in the refraction,

which, however, in the present state of the tables, will usually be very small.

If the latitude is greater than 45°, and the star's declination less than 45°, the upper transit occurs on the opposite side of the zenith from the pole. In that case h must still represent the distance of the star from the point of the horizon below the pole, and will exceed 90°. Thus, among the Greenwich observations we find

$$
\begin{aligned}
\text{1837 June 14, Capella} \quad h_1 &= 7° 18' 7''.94 \\
h &= 95 39 7 .91 \\
\hline
\varphi &= 51 28 37 .93
\end{aligned}
$$

163. *Meridian zenith distances of the sun observed near the summer and winter solstices.*—When the place of observation is near the equator, the lower culminations of stars can no longer be observed, and, consequently, the method of the preceding article cannot be used. The latitude found from stars observed at their upper culminations only is dependent upon the tabular declination, and is, therefore, subject to the error of this declination. If, therefore, an observatory is established on or near the equator, and its latitude is to be fixed independently of observations made at other places, the meridian zenith distances of stars cannot be employed. The only independent method is then by meridian observations of the sun near the solstices.

Let us at first suppose that the observations can be obtained exactly at the solstice, and the obliquity (ε) of the ecliptic is constant. The declination of the sun at the summer solstice is $= +\varepsilon$, and at the winter solstice it is $= -\varepsilon$; hence, from the meridian zenith distances ζ and ζ' observed at these times, we should have

$$
\begin{aligned}
\varphi &= \zeta + \varepsilon \\
\varphi &= \zeta' - \varepsilon
\end{aligned}
$$

the mean of which is

$$
\varphi = \tfrac{1}{2}(\zeta + \zeta')
$$

a result dependent only upon the data furnished by the observations.

Now, the sun will not, in general, pass the meridian of the observer at the instant of the solstice, or when the declination is at its maximum value ε; nor is the obliquity of the ecliptic constant. But the *changes* of the declination near the solstices are very small, and hence are very accurately obtained from the

solar tables (or from the Ephemeris which is based on these tables), notwithstanding small errors in the absolute value of the obliquity. The small change in the obliquity between two solstices is also very accurately known. If then $\Delta\varepsilon$ is the unknown correction of the tabular obliquity, and the tabular values at the two solstices are ε and ε', the true values are $\varepsilon + \Delta\varepsilon$ and $\varepsilon' + \Delta\varepsilon$; and if the tabular declinations at two observations near the solstices are $\varepsilon - x$ and $-(\varepsilon' - x')$, the true declinations will be $\delta = \varepsilon + \Delta\varepsilon - x$ and $\delta' = -(\varepsilon' + \Delta\varepsilon - x')$, and by the formula $\varphi = \zeta + \delta$ we shall have for the two observations

$$\varphi = \zeta + \varepsilon + \Delta\varepsilon - x$$
$$\varphi = \zeta' - \varepsilon' - \Delta\varepsilon + x'$$

the mean of which is

$$\varphi = \tfrac{1}{2}(\zeta + \zeta') + \tfrac{1}{2}(\varepsilon - \varepsilon') - \tfrac{1}{2}(x - x')$$

a result which depends upon the small changes $\varepsilon - \varepsilon'$ and $x - x'$, both of which are accurately known.

It is plain that, instead of computing these changes directly, it suffices to deduce the latitude from a number of observations near each solstice by employing the apparent declinations of the solar tables or the Ephemeris; then, if φ' is the mean value of the latitude found from all the observations at the northern solstice, and φ'' the mean from all at the southern solstice, the true latitude will be

$$\varphi = \tfrac{1}{2}(\varphi' + \varphi'')$$

Every observation should be the mean of the observed zenith distances of both the upper and the lower limb of the sun, in order to be independent of the tabular semidiameter and to eliminate errors of observation as far as possible.

SECOND METHOD.—BY A SINGLE ALTITUDE AT A GIVEN TIME.

164. At the instant when the altitude is observed, the time is noted by the clock. The clock correction being known, we find the true local time, and hence the star's hour angle, by the formula

$$t = \Theta - \alpha$$

in which Θ is the sidereal time and α the star's right ascension.

limb observed with the sextant and artificial horizon was 114°
40′ 30″ at 4ʰ 27ᵐ 15ˢ by a Greenwich Chronometer, which was
fast 2ᵐ 30ˢ. Index Correction of Sextant $= - 1′$ 12″, Barom.
29.72 inches, Att. Therm. 61° F., Ext. Therm. 61° F. Required
the true latitude.

Sextant reading $=$ 114° 40′ 30″		Chronometer	4ʰ 21ᵐ 15ˢ		
Index corr. $= -$ 1 12		Correction	$-$ 2 30		
114 39 18		Gr. date, March 27,	4 18 45		
App. alt. ☉ $=$ 57 19 39		Longitude $=$	2 52 56		
Semidiameter $= +$ 16 3		Local mean t. $=$	1 25 49		
Ref. and par. $= -$ 31		Eq. of time $= -$	5 19		
$h =$ 57 35 11		App. time, $t =$	1 20 30		
$\delta = +$2 51 30		$=$	20° 7′ 30″		

$$\log \sec t \quad 0.027360$$
$$\log \tan \delta \quad 8.698351$$
$$\log \tan D \quad 8.725711$$

$$\log \text{cosec } \delta \ 1.302190$$
$$\log \sin D \quad 8.725098$$
$$\log \sin h \quad 9.926445$$
$$\log \cos \gamma \quad 9.953733$$

$$D = +\ 3° 2′ 38″$$
$$\gamma = \quad 25\ 58\ 49$$
$$D - \gamma = \varphi = -\ 22\ 56\ 11$$

EXAMPLE 2.—1856 Aug. 22; suppose the true altitude of
Fomalhaut is found to be 29° 10′ 0″ when the local sidereal time
is 21ʰ 49ᵐ 44ˢ; what is the latitude?

We have $a = 22$ʰ49ᵐ44ˢ, whence $t = - 1$ʰ0ᵐ0ˢ; $\delta = - 30° 22′ 47″.5$;
$D = - 31° 15′ 13″$, $\gamma = \pm 60° 0′ 6″$, $\varphi = + 28° 44′ 53″$. The nega-
tive value of γ here gives $\varphi = - 91° 15′ 19″$; which is inadmissible.

165. The observation of equal altitudes east and west of the
meridian may be used not only for determining the time (Art.
139), but also the latitude. For the half elapsed sidereal time
between two such altitudes of a fixed star is at once the hour
angle required in the method of the preceding article. When
the sun is used in this way, the half difference between the
apparent times of the observations is the hour angle, and the
declination must be taken for noon, or more strictly for the
mean of the times of observation. By thus employing the
mean of the A.M. and P.M. hour angles and the mean of the
corresponding declinations, we obtain sensibly the same result

at the same time with that of errors of altitude and time, we should select a star as near the pole as possible, and observe it at or near its greatest elongation, on either side of the meridian. The proximity of the star to the pole enables us to facilitate the reduction of a series of observations, and we shall therefore treat specially of this case as our *Fourth Method* below.

167. When several altitudes not very far from the meridian are observed in succession, if we wish to use their mean as a single altitude, the correction for second differences (Art. 151) must be applied. It is, however, preferable to incur the labor of a separate reduction of each altitude, as we shall then be able to compare the several results, and to discuss the probable errors of the observations and of the final mean. When the observations are very near to the meridian, this separate reduction is readily effected, with but little additional labor, by the following method:

THIRD METHOD.—BY REDUCTION TO THE MERIDIAN WHEN THE TIME IS GIVEN.

168. *To reduce an altitude, observed at a given time, to the meridian.*— This is done in various ways.

(A.) If in the formula, employed in Art. 164,

$$\sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t = \sin h$$

we substitute

$$\cos t = 1 - 2 \sin^2 \tfrac{1}{2} t$$

it becomes

$$\sin \varphi \sin \delta + \cos \varphi \cos \delta - 2 \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t = \sin h$$

But

$$\sin \varphi \sin \delta + \cos \varphi \cos \delta = \cos (\varphi - \delta) \text{ or } \cos (\delta - \varphi)$$

Hence, if we put

$$\zeta_1 = \varphi - \delta, \qquad \text{or } \zeta_1 = \delta - \varphi$$

the above equation may be written

$$\cos \zeta_1 = \sin h + \cos \varphi \cos \delta (2 \sin^2 \tfrac{1}{2} t) \qquad (282)$$

If the star does not change its declination, ζ_1 is the zenith distance of the star at its meridian passage; and, being found by

this equation, we then have the latitude as from a meridian observation by the formula

$$\varphi = \delta + \zeta_1, \quad \text{or} \quad \varphi = \delta - \zeta_1$$

according as the zenith is north or south of the star.

When the star changes its declination, this method still holds *if we take δ for the time of observation*, as is evident from our formulæ, in which δ is the declination at the instant when the true altitude is h.

To compute the second member, a previous knowledge of the latitude is necessary. As the term $\cos \varphi \cos \delta (2 \sin^2 \frac{1}{2} t)$ decreases with t, if the observations are not too far from the meridian, the error produced by using an approximate value of φ will be relatively small, so that the latitude found will be a closer approximation than the assumed one; and if the computation be repeated with the new value, a still closer approximation may be made, and so on until the exact value is found.

This method is only convenient where the computer is provided with a table of natural sines and cosines, as well as a table of log. versed sines, or the logarithmic values of $2 \sin^2 \frac{1}{2} t$.

EXAMPLE.—Same as Example 1, Art. 164. $h = 57° 35' 11''$, $\delta = + 2° 51' 30''$, $t = 1^h 20^m 30^s$. Approximate value of $\varphi' = - 23°$.

		log $(2 \sin^2 \frac{1}{2} t)$	8.785726
		log cos φ	9.964026
nat. cos h	0.844201	log cos δ	9.999459
nat. no.	0.056132 . . . . log	8.749211
nat. cos ζ_1	0.900333		

$\zeta_1 = - 25° 47' 54''$ (zenith south of sun.)
$\delta = + \quad 2 \ 51 \ 30$
$\varphi = - 22 \ 56 \ 24$

differing but 13'' from the true value, although the assumed latitude was in error nearly 4'. Repeating the computation with $- 22° 54' 24''$ as the approximate latitude, we find $\varphi = - 22° 56' 11''$, exactly as in Art. 164.

169. (B.) We may also compute directly the reduction of the observed altitude to the meridian altitude. Putting

$$h_1 = \text{meridian altitude} = 90° - \zeta_1$$

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the formula (282) gives

$$\sin h_1 - \sin h = 2 \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t$$

But we have

$$\sin h_1 - \sin h = 2 \cos \tfrac{1}{2}(h_1 + h) \sin \tfrac{1}{2}(h_1 - h)$$

and hence

$$\sin \tfrac{1}{2}(h_1 - h) = \frac{\cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t}{\cos \tfrac{1}{2}(h_1 + h)} \qquad (283)$$

which gives the difference $h_1 - h$, or the correction of h to reduce it to h_1; but it requires in the second member an approximate value both of φ and of h_1, the latter being obtained from the assumed value of φ by the equation $h_1 = 90° - (\varphi - \delta)$; or, if the zenith is south of the star, by the equation $h_1 = 90° - (\delta - \varphi)$.

EXAMPLE.—Same as the above.

	$\delta =$	2° 51′ 30″		log sin² ½ t	8.484696
Approx.	$\varphi =$	− 23 00 00		log cos φ	9.964026
"	$\zeta_1 =$	25 51 30		log cos δ	9.999459
"	$h_1 =$	64 8 30		log sec ½ (h₁ + h)	0.312573
"	½ (h₁ + h) =	60 51 50		log sin ½ (h₁ − h)	8.760754
	$h_1 - h =$	6 36 33			
	$h =$	57 35 11	$\delta =$	2° 51′ 30″	
	$h_1 =$	64 11 44	$\zeta_1 =$	− 25 48 16	
			$\varphi =$	− 22 56 46	

This method does not approximate so rapidly as the preceding, but the objection is of little weight when the observations are very near the meridian. On the other hand, it has the great advantage of not requiring the use of the table of natural sines.

170. (C.) *Circummeridian altitudes.*—When a number of altitudes are observed very near the meridian,* they are called *circum-meridian* altitudes. Each altitude reduced to the meridian gives nearly as accurate a result as if the observation were taken on the meridian.

An approximate method of reducing such observations with the greatest ease is found by regarding the small arc ½ (h₁ − h) as sensibly equal to its sine; that is, by putting

$$\sin \tfrac{1}{2}(h_1 - h) = \tfrac{1}{2}(h_1 - h) \sin 1''$$

---

* *How near* to the meridian will be determined in Art. 175.

and taking h_1 for $\frac{1}{2} (h_1 + h)$, from which it differs very little, so that (283) may be put under the form

$$h_1 - h = \frac{\cos \varphi \cos \delta}{\cos h_1} \cdot \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''} \qquad (284)$$

The value in seconds of

$$m = \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''}$$

is given in Table V. with the argument t. If h', h'', h''', &c. are the observed altitudes (corrected for refraction, etc.); t', t'', t''', &c., the hour angles deduced from the observed clock times; m', m'', m''', &c., the values of m from the table; and we put the constant factor

$$\left. \begin{array}{c} A = \dfrac{\cos \varphi \cos \delta}{\cos h_1} = \dfrac{\cos \varphi \cos \delta}{\sin \zeta_1} \\[2mm] h_1 = h' \ + Am' \\ h_1 = h'' + Am'' \\ h_1 = h''' + Am''' \\ \&\text{c.} \end{array} \right\} \qquad (285)$$

we have

and the mean of all these equations gives

$$h_1 = \frac{h' + h'' + h''' + \text{etc.}}{n} + A \frac{m' + m'' + m''' + \&\text{c.}}{n}$$

in which n is the number of observations; or

$$h_1 = h_0 + Am_0 \qquad (286)$$

in which h_0 denotes the mean of the observed altitudes corrected for refraction, &c., and m_0 the mean of the values of m.

When h_1 has been thus found, the latitude is deduced as from any meridian altitude, only observing that for the sun the declination to be used is that which corresponds to the mean of the times of observation, as has already been remarked in Art. 168.

EXAMPLE.—At the U. S. Naval Academy, 1849 June 22, circummeridian altitudes of β *Ursae Minoris* were observed with a Troughton sextant from an artificial horizon, as in the following table. The times were noted by a sidereal chronometer which

was fast 1^m $45^s.7$. The index correction of the sextant was $- 14'$ $58''$, Barometer, 30.21 inches, Att. Therm. 75° F., Ext. Therm. 74° F.

The right ascension of the star was 14^h 51^m $14^s.0$

Chronometer fast $+ 1$ 45.7

Chronometer time of star's transit 14 52 59.7

The hour angles in the column t are found by taking the difference between each observed chronometer time and this chronometer time of transit.

2 Alt. ✳	Chronom.	t	m
108° 39' 40"	14^h 45^m 47^s.	7^m $12^s.7$	102.0
39 50	47 1.	5 58.7	70.2
40 40	48 54.5	4 5.2	82.8
41 0	51 29.5	1 30.2	4.4
41 0	54 36.5	1 36.8	5.1
40 30	56 22.	3 22.3	22.3
40 20	57 43.	4 43.3	43.7
40 0	58 47.5	5 47.8	66.0
40 0	15 0 17.5	7 17.8	104.5
39 20	2 10.	9 10.3	165.1

Mean 108 40 14 $m_0 =$ 61.61

Ind. corr. $- 14$ 58

108 25 16 Assumed $\phi =$ 38° 59' 0"

54 12 38 $\delta =$ 74 46 36 .9

Refr. $- 42$ .0 Approx. $\zeta_1 =$ 35 47 36 .9 log cos ϕ 9.8906

Am_0 $+ 21$ .5 log cos δ 9.4192

$h_1 =$ 54 12 17 .5 log cosec ζ_1 0.2329

$\zeta_1 = - 35$ 47 42 .5 log A 9.5427

$\delta =$ 74 46 36 .9 log m_0 1.7897

$\phi =$ 88 58 54 .4 log Am_0 1.3324

REMARK 1.—The reduction $h_1 - h$ increases as the denominator of A decreases, that is, as the meridian zenith distance decreases. The preceding method, therefore, as it supposes the reduction to be small, should not be employed when the star passes very near the zenith, unless at the same time the observations are restricted to very small hour angles. It can be shown, however, from the more complete formulæ to be given presently, that so long as the zenith distance is not less than 10°, the reduction computed by this method may amount to 4' 30" without being in error more than 1"; and this degree of accuracy suffices for even the best observations made with the sextant.

REMARK 2.—If in (284) we put $\sin \frac{1}{2} t = \mathcal{Y} \sin 1''$. t (t being in seconds of time), we have

$$h_1 - h = \frac{\cos \varphi \cos \delta}{\cos h_1} \cdot \frac{225}{2} \sin 1'' . t^2 = at^2 \qquad (287)$$

in which a denotes the product of all the *constant* factors. It follows from this formula that *near the meridian the altitude varies as the square of the hour angle*, and not simply in proportion to the time. Hence it is that near the meridian we cannot reduce a number of altitudes by taking their mean to correspond to the mean of the times, as is done (in most cases without sensible error) when the observations are remote from the meridian. The method of reduction above exemplified amounts to separately reducing each altitude and then taking the mean of all the results.

171. (D.) *Circummeridian altitudes more accurately reduced.*—The small correction which the preceding method requires will be obtained by developing into series the rigorous equation (282). This equation, when we put $\zeta = 90° - h =$ true zenith distance deduced from the observation, may be put under the form

$$\cos \zeta = \cos \zeta_1 - 2 \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t$$

which developed in series* gives, neglecting sixth and higher powers of $\sin \frac{1}{2} t$,

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* If we put $y = 2 \cos \phi \cos \delta \sin^2 \tfrac{1}{2} t$, the equation to be developed is

$$\cos \zeta = \cos \zeta_1 - y \qquad (a)$$

in which ζ_1 is constant and ζ may be regarded as a function of y; so that by MAC-LAURIN's Theorem

$$\zeta = fy = (f) + \left(\frac{df}{dy}\right) y + \tfrac{1}{2}\left(\frac{d^2f}{dy^2}\right) y^2 + \&c. \qquad (b)$$

in which $(f), \left(\dfrac{df}{dy}\right)$, &c. denote the values of fy and its differential coefficients when $y = 0$. The equation (a) gives, by differentiation,

$$\sin \zeta \frac{d\zeta}{dy} = 1 \qquad \frac{d\zeta}{dy} = \frac{1}{\sin \zeta}$$

$$\frac{d^2\zeta}{dy^2} = - \frac{\cos \zeta}{\sin^2 \zeta} \cdot \frac{d\zeta}{dy} = - \frac{\cot \zeta}{\sin^2 \zeta} \quad \&c.$$

$$\zeta_1 = \zeta - \frac{\cos\varphi\cos\delta}{\sin\zeta_1}\cdot\frac{2\sin^2\frac{1}{2}t}{\sin 1''} + \left(\frac{\cos\varphi\cos\delta}{\sin\zeta_1}\right)^2\cdot\frac{2\cot\zeta_1\sin^4\frac{1}{2}t}{\sin 1''} \quad (288)$$

By this formula, first given by DELAMBRE, the reduction to the meridian consists of two terms, the first of which is the same as that employed in the preceding method, and the second is the small correction which that method requires. These two terms will be designated as the "1st Reduction" and "2d Reduction." Putting

$$m = \frac{2\sin^2\frac{1}{2}t}{\sin 1''} \qquad\qquad n = \frac{2\sin^4\frac{1}{2}t}{\sin 1''}$$

$$A = \frac{\cos\varphi\cos\delta}{\sin\zeta_1} \qquad\qquad B = A^2\cot\zeta_1$$

we have

$$\zeta_1 = \zeta - Am + Bn \quad (289)$$

If a number of observations are taken, we have a number of equations of this form, the mean of which will be

$$\zeta_1 = \zeta_0 - Am_0 + Bn_0$$

in which ζ_0 is the arithmetical mean of the observed zenith distances, m_0 and n_0 the arithmetical means of the values of m and n corresponding to the values of t. The values of n are also given in Table V.

Having found ζ_1, we have the latitude, as before, by the formula

$$\varphi = \delta + \zeta_1$$

in which we must give ζ_1 the negative sign when the zenith is south of the star, and it must be remembered that for the sun (or any object whose proper motion is sensible) δ must be the mean of the declinations belonging to the several observations,

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But when $y = 0$ we have, by (a), $\zeta = \zeta_1$, so that (b) becomes

$$\zeta = \zeta_1 + \frac{y}{\sin\zeta} - \frac{y^2\cot\zeta_1}{2\sin^2\zeta_1} + \frac{1}{6}(1 + 3\cot^2\zeta_1)\frac{y^3}{\sin^3\zeta_1} - \&c. \quad (c)$$

Restoring the value of y, this gives the development used in the text, observing that as ζ and ζ_1 are supposed to be in seconds of arc, the terms of the series are divided by $\sin 1''$ to reduce them to the same unit.

or, which is the same, the declination corresponding to the mean of the times of observation.*

Finally, if the star is near the meridian below the pole, the hour angles should be reckoned from the instant of the lower transit. Recurring to the formula

$$\cos \zeta = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t$$

in which t is the hour angle reckoned from the upper transit, we observe that if this angle is reckoned from the lower transit we must put $180° - t$ instead of t, or $- \cos t$ for $+ \cos t$, and then we have

$$\cos \zeta = \sin \varphi \sin \delta - \cos \varphi \cos \delta \cos t$$

and, substituting as before,

$$\cos t = 1 - 2 \sin^2 \tfrac{1}{2} t$$

this gives

$$\cos \zeta = - \cos (\varphi + \delta) + 2 \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t$$

or, since for lower culminations we have $\zeta_1 = 180° - (\varphi + \delta)$ and $\cos \zeta_1 = - \cos (\varphi + \delta)$,

$$\cos \zeta = \cos \zeta_1 + 2 \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t$$

which developed gives

$$\zeta_1 = \zeta + \frac{\cos \varphi \cos \delta}{\sin \zeta_1} \cdot \frac{2 \sin^2 \tfrac{1}{2} t}{\sin 1''} + \left(\frac{\cos \varphi \cos \delta}{\sin \zeta_1} \right)^2 \cdot \frac{2 \cot \zeta_1 \sin^4 \tfrac{1}{2} t}{\sin 1''}$$

or

$$\zeta_1 = \zeta + Am + Bn \text{ (sub polo)} \qquad (290)$$

which is computed by the same table, but both first and second reductions here have the same sign.

If a star is observed with a sidereal chronometer the daily rate of which is so small as to be insensible during the time of

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* To show that the mean declination is to be used, we may observe that for each observation we have put $\zeta_1 = \varphi - \delta$, and that if δ', δ'', &c., are the several declinations, the several equations of the form (289) will give

$$\varphi = \delta' + \zeta' - Am' + A^2 \cot \zeta_1 n'$$
$$\varphi = \delta'' + \zeta'' - Am'' + A^2 \cot \zeta_1 n''$$
$$\&c.,$$

the mean of which, if $\delta =$ mean of δ', δ'', &c., will be

$$\varphi = \delta + \zeta_0 - Am_0 + A^2 \cot \zeta_1 n_0 = \delta + \zeta_1$$

the observations, the hour angles t are found by merely taking the difference between each noted time and the chronometer time of the star's transit, as in the example of Article 170. But if we wish to take account of the rate of the chronometer, it can be done without separately correcting each hour angle, as follows: Let δT be the rate of the chronometer in 24^h (δT being positive for losing rate, Art. 137); then, if t is the hour angle given directly by the chronometer, and t' the true hour angle, we have

$$t' : t = 24^h : 24^h - \delta T = 86400^s : 86400^s - \delta T$$

whence

$$t' = t \cdot \left[\frac{1}{1 - \dfrac{\delta T}{86400}} \right]$$

Instead of $\sin \frac{1}{2} t$ we must use $\sin \frac{1}{2} t'$; for which we shall have, with all requisite precision,

$$\sin \tfrac{1}{2} t' = \sin \tfrac{1}{2} t \cdot \frac{t'}{t}, \text{ or } \sin^2 \tfrac{1}{2} t' = \sin^2 \tfrac{1}{2} t \cdot \left(\frac{t'}{t} \right)^2$$

Hence, if we put

$$k = \left[\frac{1}{1 - \dfrac{\delta T}{86400}} \right]^2 = \left(\frac{t'}{t} \right)^2$$

we shall have

$$Am = k \cdot \frac{\cos \varphi \cos \delta}{\sin \zeta_1} \cdot \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''}$$

so that if we compute A by the formula

$$A = k \cdot \frac{\cos \varphi \cos \delta}{\sin \zeta_1}$$

we can take $m = \dfrac{2 \sin^2 \frac{1}{2} t}{\sin 1''}$ for the actual chronometer intervals, and no further attention to the rate is required.

The factor k can be given in a small table with the argument "rate," in connection with the table for m, as in our Table V.

If a star is observed with a mean time chronometer, the intervals are not only to be corrected for rate, but also to be reduced

from mean to sidereal intervals by multiplying them by $\mu =$ 1.00273791 (Art. 49); so that for $\sin^2 \frac{1}{2} t$ we must substitute $k \sin^2$ $(\frac{1}{2} . \mu t)$, or, with sufficient precision, $k \mu^2 \sin^2 \frac{1}{2} t$.

If the sun is observed with a mean time chronometer, the intervals are both to be corrected for rate and reduced from mean solar to apparent solar intervals. The mean interval differs from the apparent only by the change in the equation of time during the interval, and this change may be combined with the rate of the chronometer. Denoting by δE the *increase* of the equation of time in 24^h (remembering that E is to be regarded as positive when it is additive to apparent time), and by δT the rate of the chronometer on mean time, we may regard $\delta T - \delta E$ as the rate of the chronometer on apparent time. Instead of the factor k we shall then have a factor k', which is to be found by the formula

$$k' = \left[\frac{1}{1 - \dfrac{\delta T - \delta E}{86400}} \right]^2$$

which may be taken from the table for k by taking $\delta T - \delta E$ as the argument.

Finally, if the sun is observed with a sidereal chronometer, we must multiply $\sin^2 \frac{1}{2} t$ not only by k' but by $\dfrac{1}{\mu^2}$.

Denoting μ^2 by i and $\dfrac{1}{\mu^2}$ by i', these rules may be collected, for the convenience of reference, as follows:

Star by sidereal chron., $A = k \cdot \dfrac{\cos \varphi \cos \delta}{\sin \zeta_1}$

Star by mean time chron., $A = k i \cdot \dfrac{\cos \varphi \cos \delta}{\sin \zeta_1} [\log i = 0.002375]$ (291)

Sun by mean time chron., $A = k' \cdot \dfrac{\cos \varphi \cos \delta}{\sin \zeta_1}$

Sun by sidereal chron., $A = k' i' \dfrac{\cos \varphi \cos \delta}{\sin \zeta_1} [\log i' = 9.997625]$ (292)

for which $\log k$ will be taken from Table V. with the argument rate of the chronometer $= \delta T$; and $\log k'$ from the same table

with the argument $\delta T - \delta E =$ daily rate of the chronometer diminished by the daily increase of the equation of time.

EXAMPLE.—1856 March 15, at a place assumed to be in latitude 37° 49′ N. and longitude 122° 24′ W., suppose the following zenith distances of the sun's lower limb to have been observed with an Ertel universal instrument,[*] Barom. 29.85 inches, Att. Therm. 65° F., Ext. Therm. 63° F. The chronometer, regulated to the local mean time, was, at noon, slow $11^m 20^s.8$, with a daily losing rate of $6^s.6$.

Obs'd zen. dist.	Chronometer.	t	m	n
40° 8′ 40″.7	23^h 37^m 35^s.	−19^m 58^s.8	783″.3	1″.49
40 2 16 .5	42 3 .	−15 30 .8	472 .4	0 .54
39 57 28 .3	46 29 .5	−11 4 .3	240 .6	0 .14
39 54 17 .2	50 46 .5	− 6 47 .3	90 .5	0 .02
39 52 33 .	55 16 .	− 2 17 .8	10 .4	0 .00
39 52 34 .5	0 0 37 .5	+ 3 3 .7	18 .4	0 .00
39 54 28 .6	5 13 .	7 39 .2	115 .0	0 .03
39 58 9 .8	9 49 .5	12 15 .7	295 .1	0 .21
40 3 0 .3	14 8 .	16 34 .2	538 .9	0 .70
40 9 36 .	18 31 .	20 57 .2	861 .4	1 .80

Means 39 59 18 .5 $t_0 = +\ 0\ 29.1$ $m_0 = 342\ .60$ $n_0 = 0\ .49$

The equation of time at the local noon being $+ 8^m 54^s.6$, we have

Mean time of app. noon $=\ \ 0^h\ 8^m\ 54^s.6$
Chronometer slow $=\ \ \ \ 11\ 20\ .8$
Chr. time of app. noon $= 23\ 57\ \ 33\ .8$

The difference between this and the observed chronometer times gives the hour angles t as above.

The mean of the hour angles being $+ 29^s.1$, the declination is to be taken for the local apparent time $0^h 0^m 29^s.1$, or for the Greenwich mean time March 15, $8^h 18^m 59^s.7$; whence

$$\delta = -\ 1° 48′ 8″.8$$
$$\text{(Approximate)}\ \varphi = +\ 37\ 49\ 0\ .$$
$$\text{``}\qquad \zeta_1 = \ \ \ 39\ 37\ 8\ .8$$

The increase of the equation of time in 24^h is $\delta E = -\ 17^s.4,$

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[*] See Vol. II., *Altitude and Azimuth Instrument*, for the method of observing the zenith distances.

and, the chronometer rate being $\delta T = + 6^s.6$, we have $\delta T - \delta E = + 24^s.0$, with which as the argument "rate" in Table V. we find $\log k' = 0.00024$.

The computation of the latitude is now carried out as follows:

$\log \cos \varphi$	9.89761	Mean observed zen. dist. $\bigodot =$		$39°$	$59'$	$18''.5$	
$\log \cos \delta$	9.99979		$r - p =$	$+$		41	$.8$
$\log \operatorname{cosec} \zeta_1$	0.19540	$\log A^2$	0.1861	$S =$	$-$	16	6	$.5$
$\log k'$	0.00024	$\log \cot \zeta_1$	0.0821	$A m_0 =$	$-$	7	4	$.4$
$\log A$	0.09304	$\log B$	0.2682	$B n_0 =$	$+$		0	$.9$
$\log m_0$	2.53479	$\log n_0$	9.6902	$\zeta_1 =$	39	36	50	$.3$
$\log A m_0$	2.62783	$\log B n_0$	9.9584	$\delta =$	-1	48	8	$.8$
				$\varphi =$	37	48	41	$.5$

The assumed value of φ being in error, the value of A is not quite correct; but a repetition of the computation with the value of φ just found does not in this case change the result so much as $0''.1$.

172. (E.) *Gauss's method of reducing circummeridian altitudes of the sun.*—The preceding method of reduction is both brief and accurate, and the latitude found is the mean of all the values that would be found by reducing each observation separately. This separate reduction, however, is often preferred, notwith-standing the increased labor, as it enables us to compare the observations with each other, and to discuss the probable error of the final result; and it is also a check against any gross error. But, if we separately reduce the observations by the preceding method, we must not only interpolate the refraction for each altitude, but also the declination for each hour angle. GAUSS proposed a method by which the latter of these interpolations is avoided. He showed that if we reckon the hour angles, not from apparent noon, but from *the instant when the sun reaches its maximum altitude*, we can compute the reduction by the method above given, and use the *meridian declination* for all the observa-tions. This method is, indeed, not quite so exact as the preced-ing; but I shall show how it may be rendered quite perfect in practice by the introduction of a small correction.

In the rigorous formula

$$\cos \zeta = \sin \varpi \sin \delta + \cos \varphi \cos \delta \cos t$$

δ is the declination corresponding to the hour angle t. If then

$\Delta\delta =$ the hourly increase of the declination, *positive* when the sun is moving *northward*,

$\delta_1 =$ the declination at noon,

and if t is expressed in seconds of time, we have

$$\delta = \delta_1 + \frac{t \cdot \Delta\delta}{3600} = \delta_1 + x$$

where, since $\Delta\delta$ never exceeds $60''$, x will not exceed $30''$ so long as $t < 30^m$. Hence we may substitute, with great accuracy,

$$\sin \delta = \sin \delta_1 + \cos \delta_1 \sin x$$
$$\cos \delta = \cos \delta_1 - \sin \delta_1 \sin x$$

and the above formula becomes

$$\cos \zeta = \sin \varphi \sin \delta_1 + \cos \varphi \cos \delta_1 \cos t + \sin (\varphi - \delta_1) \sin x$$
$$+ 2 \cos \varphi \sin \delta_1 \sin^2 \tfrac{1}{2} t \sin x$$

The last term is extremely small, rarely affecting the value of ζ by as much as $0''.1$; and since x is proportional to the hour angle, and therefore has opposite signs for observations on different sides of the meridian, the effect of this term will nearly or quite disappear from the mean of a series of observations properly distributed before and after the meridian passage. Now, we have

$$\sin x = \frac{t \Delta\delta \sin 1''}{3600} = 15 t \sin 1'' \cdot \frac{\Delta\delta}{54000}$$

Let

$$\sin \vartheta = \frac{\Delta\delta}{54000} \cdot \frac{\sin (\varphi - \delta_1)}{\cos \varphi \cos \delta_1}$$

then, taking

$$15 t \sin 1'' = \sin t + \tfrac{1}{6} \sin^3 t$$

we have

$$\sin x = (\sin t + \tfrac{1}{6} \sin^3 t) \sin \vartheta \cdot \frac{\cos \varphi \cos \delta_1}{\sin (\varphi - \delta_1)}$$

and the formula for $\cos \zeta$ becomes, by omitting the last term,

$$\cos \zeta = \sin \varphi \sin \delta_1 + \cos \varphi \cos \delta_1 (\cos t + \sin t \sin \vartheta)$$
$$+ \tfrac{1}{6} \cos \varphi \cos \delta_1 \sin^3 t \sin \vartheta$$

from the instant the sun reaches its maximum altitude, we have

$$\zeta_1 = \zeta - Am + Bn \qquad (293)$$

Since ζ_1 differs from the latitude by the constant quantity δ', its value found from each observation should be the same. Taking its mean value, we have

$$\varphi = \zeta_1 + \delta'$$

The angle ϑ, being very small, may be found with the utmost precision by the formula

$$\vartheta = \frac{1}{810000 \sin 1''} \cdot \frac{\Delta\delta}{A} = [9.40594] \frac{\Delta\delta}{A} \qquad (294)$$

which gives ϑ in seconds of the chronometer when A has been computed by the formula (292).

The most simple method of finding the corrected hour angles t' will be to add ϑ to the chronometer time of apparent noon, and then take the difference between this corrected time and each observed time.

If we put $\delta' = \delta_1 + y$, we have

$$y = A \cdot \frac{2 \sin^2 \tfrac{1}{2} \vartheta}{\sin 1''} \qquad (295)$$

which requires only one new logarithm to be taken, namely, the value of log m from Table VI. with the argument ϑ. We then have, finally,

$$\varphi = \zeta_1 + \delta_1 + y \qquad (296)$$

EXAMPLE.—The same as that of the preceding article. We have there employed the assumed latitude 37° 49′; but, since even a rough computation of two or three observations will give a nearer value, let us suppose we have found as a first approximation $\varphi = 37°$ 48′ 45″. With this and the meridian declination $\delta_1 = -1°$ 48′ 9″.2, and log $k' = 0.00024$ as before, we now find, by (292),

$$\log A = 0.09310 \qquad \log B = 0.2683$$

We have also there found the chronometer time of apparent

noon $= 23^h 57^m 33^s.8$. We now take from the Ephemeris $\Delta\delta = + 59''.22$, and hence, by (294),

$$
\begin{array}{rl}
\text{log } \Delta\delta & 1.7725 \\
\text{ar. co. log } A & 9.9070 \\
\text{const. log} & 9.4059 \\
\hline
\vartheta = + 12^s.2 \qquad \text{log } \vartheta & 1.0854
\end{array}
$$

Hence the chronometer time of the maximum altitude is $23^h 57^m 33^s.8 + 12^s.2 = 23^h 57^m 46^s$, which gives the hour angles t' as below:

t'	log m	log Am	log n	log Bn
$-20^m 11^s.$	2.90274	2.99584	0.1900	0.4583
15 43.	2.68559	2.77869	9.7657	0.0340
11 16.5	2.39719	2.49029	9.1801	9.4484
6 59.5	1.98216	2.07526	8.3010	8.5693
$-$ 2 30.	1.08891	1.18201		
$+$ 2 51.5	1.20525	1.29835		
7 27.	2.03780	2.13040	8.4771	8.7454
12 3.5	2.45551	2.54861	9.2943	9.5626
16 22.	2.72077	2.81387	9.8261	0.0944
20 45.	2.92677	3.01987	0.2381	0.5064

The refraction may be computed from the tables first for a mean zenith distance, and then with its variation in one minute (which will be found with sufficient accuracy from the table of mean refraction) its value for each zenith distance is readily found. The parallax, which is here sensibly the same $(= 5''.54)$ for all the observations, is subtracted from the refraction, and the results are given in the column $r - p$ of the following computation. The numbers in the 3d and 4th columns are found from their logarithms above; and the last column contains the several values of the minimum zenith distance of the sun's lower limb, formed by adding together the numbers of the preceding columns. To the mean of these we then apply the sun's semidiameter, the meridian declination, and the correction y, which are all constant for the whole series of observations.

Obs'd ζ	$r - p$	Am	Bn	ζ_1
40° 8' 40".7	+ 42".1	− 16' 30".5	+ 2".9	39° 52' 55".2
40 2 16 .5	41 .9	10 0 .7	1 .1	58 .8
39 57 28 .3	41 .8	5 9 .2	0 .3	61 .2
39 54 17 .2	41 .7	1 58 .9	0 .0	60 .0
39 52 33 .	41 .6	0 15 .2	0 .0	59 .4
39 52 34 .5	41 .6	0 19 .9	0 .0	56 .2
39 54 28 .6	41 .7	2 15 .0	0 .1	55 .4
39 58 9 .8	41 .8	5 53 .7	0 .4	58 .3
40 3 0 .3	41 .9	10 51 .4	1 .2	52 .0
40 9 36 .	42 .1	17 26 .8	3 .2	54 .5

(Lower limb) Mean $\zeta_1 =$ 39 52 57 .10

$\log \dfrac{2 \sin^2 \frac{1}{2} \vartheta}{\sin 1'}$ 8.9090

$\log A$ 0.0931

$\log y$ 9.0021

Semidiameter $= −$ 16 6 .49

$\delta_1 = − 1$ 48 9 .20

$y = +$ 0 .10

$\varphi =$ 37 48 41 .51

This result agrees precisely with that found before. If we suppose all the observations to be of the same weight, we can now determine the probable error of observation. Denoting the difference between each value of ζ_1 and the mean of all by v, and the sum of the squares of v by $[vv]$, according to the notation used in the method of least squares, we have

v	vv
− 1".9	3.61
+ 1 .7	3.24
+ 4 .1	16.81
+ 2 .9	8.41
+ 2 .3	5.29
− 0 .9	.81
− 1 .7	2.89
+ 1 .2	1.44
− 5 .1	26.01
− 2 .6	6.76

$n = 10,\ [vv] = 75.27$

Mean error of a single observation $= \sqrt{\dfrac{[vv]}{n-1}} = 2".89$

Mean error of the final value of

$\varphi = \dfrac{2.89}{\sqrt{10}} = 0".91$

Probable error of a single obs. $= 2".89 \times 0.6745 = 1".95$

" " of φ $= 0 .91 \times 0.6745 = 0 .62$

It must not be forgotten that the probable error 1".95 here represents the probable error of *observation* only: a constant error of the instrument, affecting all the observations, will form no part of this error; and the same is true of an error in the refraction.

173. For the most refined determinations of the latitude, standard stars are to be preferred to the sun. Their declinations are somewhat more precisely known; the instrument is in night observations less liable to the errors produced by changes of temperature during the observations; constant instrumental errors and errors of refraction may be eliminated to a great extent by combining north and south stars; or errors of declination may be avoided by employing only circumpolar stars at or near their upper and lower culminations. In general, errors of circummeridian altitudes are eliminated under the same conditions as those of meridian observations, since the former are reduced to the meridian with the greatest precision. See the next following article.

For a great number of nice determinations of the latitude by circummeridian altitudes of stars north and south of the zenith and of circumpolar stars, see PUISSANT, *Nouvelle Description Géométrique de la France.*

174. *Effect of errors of zenith distance, declination, and time, upon the latitude found by circummeridian altitudes.*—Differentiating (289), regarding A as constant, and neglecting the variations of the last term, which is too small to be sensibly affected by small errors of t, we have, since $d\varphi = d\zeta_1 + d\delta$,

$$d\varphi = d\zeta + d\delta - \frac{2\,A\,\sin t}{\sin 1''}\,(15\,dt)$$

The errors $d\zeta$ and $d\delta$ affect the resulting latitude by their whole amount. The coefficient of dt has opposite signs for east and west hour angles; and therefore, if the observations are so taken as to consist of a number of pairs of equal zenith distances east and west of the meridian, a small constant error in the hour angles, arising from an imperfect clock correction, will be eliminated in the mean. This condition is in practice nearly satisfied when the same number of observations are taken on each side of the meridian, the intervals of time between the successive observations being made as nearly equal as practicable.

An error in the assumed latitude which affects A is eliminated by repeating the computation with the latitude found by the first computation. An error in the declination which would affect A is not to be supposed.

175. *To determine the limits within which the preceding methods of reducing circummeridian altitudes are applicable.—First.* In the method of Art. 170 we employ only the " first reduction" ($= Am$), which is the first term of the more complete reduction expressed by (288). The error of neglecting the " second reduction" ($= Bn$) increases with the hour angle, and if this method is to be used it becomes necessary to determine the value of the hour angle at which this reduction would be sensible. We have

$$ Bn = A^2 \cot \zeta_1 \frac{2 \sin^4 \tfrac{1}{2} t}{\sin 1''} $$

whence if we put b for Bn and

$$ F = \sqrt{\tfrac{1}{2} \sin 1'' \tan \zeta_1} $$

we derive

$$ \sin^2 \tfrac{1}{2} t = \frac{F}{A} \sqrt{b} \tag{298} $$

Since $\zeta_1 = \varphi - \delta$, F and A are but functions of φ and δ; and therefore by this formula we can compute the values of t for any assigned value of b, and for a series of values of φ and δ. Table VII.A gives the values of t in minutes computed by (298) when $b = 1''$. That is, calling t_1 the tabular hour angle and t the hour angle for any assigned limit of error b, we have

$$ \sin^2 \tfrac{1}{2} t_1 = \frac{F}{A} \qquad\qquad \sin^2 \tfrac{1}{2} t = \sin^2 \tfrac{1}{2} t_1 \sqrt{b} $$

As the limits are not required with great precision, we may substitute for the last equation the following:

$$ t = t_1 \sqrt[4]{b} $$

If we take $b = 0''.1$, we have $\sqrt[4]{b} = 0.56$, or nearly $\tfrac{1}{2}$: hence *the limiting hour angle at which the second reduction amounts to $0''.1$ is about $\tfrac{1}{2}$ the angle given in* Table VII.A.

EXAMPLE.—How far from the meridian may the observations in the example p. 237 be extended before the error of the method of reduction there employed amounts to $1''$? With $\varphi = +39°$, $\delta = +75°$, Table VII.A gives $t_1 = 30^m$. Hence

rapidly with the zenith distance; and hence in general very small zenith distances are to be avoided. But the observations may be extended somewhat beyond the limits of our tables, since the errors which affect only the extreme observations are reduced in taking the mean of a series.

FOURTH METHOD.—BY THE POLE STAR.

176. The latitude may be deduced with accuracy from an altitude of the pole star observed at any time whatever, when this time is known. The computation may be performed by (281); but when a number of successive observations are to be reduced, the following methods are to be preferred. If we put

$$p = \text{the star's polar distance,}$$

we have, by (14),

$$\sin h = \sin \varphi \cos p + \cos \varphi \sin p \cos t$$

in which the hour angle t and the altitude h are derived from observation and φ is the required latitude. Now, p being small (at present less than $1° 30'$), we may develop φ in a series of ascending powers of p, and then employ as many terms as we need to attain any given degree of precision. The altitude cannot differ from the latitude by more than p: if, then, we put

$$\varphi = h - x$$

x will be a small correction of the same order of magnitude as p. We have[*]

$\sin \varphi = \sin (h - x) = \sin h - x \cos h - \frac{1}{2} x^2 \sin h + \frac{1}{6} x^3 \cos h + \&\text{c.}$
$\cos \varphi = \cos (h - x) = \cos h + x \sin h - \frac{1}{2} x^2 \cos h - \frac{1}{6} x^3 \sin h + \&\text{c.}$
$\sin p = p - \frac{1}{6} p^3 + \&\text{c.}$
$\cos p = 1 - \frac{1}{2} p^2 + \&\text{c.}$

which substituted in the above formula for $\sin h$ give

$\sin h = \sin h - x \cos h + p \cos t \cos h - \frac{1}{2} (x^2 - 2 xp \cos t + p^2) \sin h + \&\text{c.}$

and from this we obtain the following general expression of the correction:

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[*] Pl. Trig. (408) and (406).

$$x = p \cos t - \tfrac{1}{2} (x^2 - 2\,xp \cos t + p^2) \tan h$$
$$+ \tfrac{1}{6} (x^3 - 3\,x^2 p \cos t + 3\,xp^2 - p^3 \cos t)$$
$$+ \tfrac{1}{24} (x^4 - 4\,x^3 p \cos t + 6\,x^2 p^2 - 4\,xp^3 \cos t + p^4) \tan h$$
$$- \&c. \tag{a}$$

For a first approximation, we take

$$x = p \cos t \tag{b}$$

and, substituting this in the second term of (a), we find for a second approximation, neglecting the third powers of p and x,

$$x = p \cos t - \tfrac{1}{2} p^2 \sin^2 t \tan h \tag{c}$$

Substituting this value in the second and third terms of (a), we find, as a third approximation,

$$x = p \cos t - \tfrac{1}{2} p^2 \sin^2 t \tan h + \tfrac{1}{3} p^3 \cos t \sin^2 t \tag{d}$$

This value, substituted in the second, third, and fourth terms of (a), gives, as a fourth approximation,

$$x = p \cos t - \tfrac{1}{2} p^2 \sin^2 t \tan h + \tfrac{1}{3} p^3 \cos t \sin^2 t - \tfrac{1}{8} p^4 \sin^4 t \tan^3 h$$
$$+ \tfrac{1}{24} p^4 (4 - 9 \sin^2 t) \sin^2 t \tan h \tag{e}$$

In these formulæ, to obtain x in seconds when p is given in seconds, we must multiply the terms in p^2, p^3, and p^4 by $\sin 1''$, $\sin^2 1''$, $\sin^3 1''$, respectively.

In order to determine the relative accuracy of these formulæ, let us examine the several terms of the last, which embraces all the others. The value of t, which makes the last term of (e) a maximum, will be found by putting the differential coefficient of $(4 - 9 \sin^2 t) \sin^2 t$ equal to zero; whence

$$4 \sin t \cos t \, (2 - 9 \sin^2 t) = 0$$

which is satisfied by $t = 0$, $t = 90°$, or $\sin^2 t = \tfrac{2}{9}$, the last of which alone makes the second differential coefficient negative. The maximum value of the term is, then, $\tfrac{1}{54} p^4 \sin^3 1'' \tan h$, which for $p = 1° 30' = 5400''$ is $0''.0018 \tan h$. This can amount to $0''.01$ only when h is nearly 80°,—that is, when the latitude is nearly 80°. This term, therefore, is wholly inappreciable in every practical case.

The term $\frac{1}{3}p^4 \sin^3 1'' \sin^4 t \tan^3 h$ is a maximum for $\sin t = 1$, in which case, for $p = 5400''$, it is $0''.0121 \tan^3 h$. This amounts to $0''.1$ when $h = 64°$, and to $1''$. when $h = 77°$.

For the maximum of the term $\frac{1}{3} p^3 \sin^2 1'' \cos t \sin^2 t$ we have, by putting the differential coefficient of $\cos t \sin^2 t$ equal to zero,

$$\sin t \, (2 - 3 \sin^2 t) = 0$$

which gives $\sin^2 t = \frac{2}{3}$, and consequently $\cos t = \sqrt{\frac{1}{3}}$; and hence the maximum value of this term is $\frac{1}{3} p^3 \sin^2 1'' \sqrt{\frac{1}{3}} = 0''.475$. Since the maximum values of this and the following terms do not occur for the same value of t, their aggregate will evidently never amount to $1''$ in any practical case.

Hence, *to find the latitude by the pole star within* $1''$, we have the formula

$$\varphi = h - p \cos t + \tfrac{1}{2} p^2 \sin 1'' \sin^2 t \tan h \qquad (300)$$

The hour angle t is to be deduced from the sidereal time Θ of the observation and the star's right ascension α, by the formula

$$t = \Theta - \alpha$$

To facilitate the computation of the formula (300), tables are given in every volume of the British Nautical Almanac and the Berlin Jahrbuch; but the formula is so simple that a direct computation consumes very little more time than the use of these tables, and it is certainly more accurate.

EXAMPLE.—(From the Nautical Almanac for 1861).—On March 6, 1681, in Longitude 37° W., at $7^h 43^m 35^s$ mean time, suppose the altitude of *Polaris*, when corrected for the error of the instrument, refraction, and dip of the horizon, to be 46° 17' 28'': required the latitude.

Mean time		$7^h 43^m 35^s$.	
Sid. time mean noon, March 6,		22 56	47.9
Reduction for $7^h 43^m 35^s$		1	16.2
Reduction for Long. $2^h 28^m$			24.3
Sidereal time	$\Theta =$	6 42	3.4
March 6, $p = 1° 25' 32''.7$	$\alpha =$	1 7	39.0
	$t =$	5 34	24.4
		$= 83° 36' 6''$	

Hence, by formula (300),

log p	3.71035	log p^2	7.4207
log cos t	9.04704	log sin^2 t	9.9946
log 1st term	2.75739	log tan h	0.0196
		log $\frac{1}{3}$ sin 1″	4.3845
		log 2d term	1.8194

$$h = 46° \ 17' \ 28''$$
$$\text{1st term} = - \quad 9 \ 32 \ .0$$
$$\text{2d } \quad = + \quad 1 \ \ 6 \ .0$$
$$\varphi = 46 \quad 9 \ \ 2 \ .0$$

By the Tables in the Almanac, $\varphi = 46° \ 9' \ 1''$

177. If we take all the terms of (e) except the last, which we have shown to be insignificant, we have the formula

$$\varphi = h - p \cos t + \tfrac{1}{2} p^2 \sin 1'' \sin^2 t \tan h$$
$$- \tfrac{1}{3} p^3 \sin^2 1'' \cos t \sin^2 t + \tfrac{1}{8} p^4 \sin^3 1'' \sin^4 t \tan^3 h \qquad (301)$$

which is exact within $0''.01$ for all latitudes less than 75°, and serves for the reduction of the most refined observations with first-class instruments.

If we have taken a number of altitudes in succession, the separate reduction of each by this formula will be very laborious. To facilitate the operation, PETERSEN has computed very convenient tables, which are given in SCHUMACHER's *Hülfstafeln*, edited by WARNSTORFF. These tables give the values of the following quantities:

$$\alpha = p_0 \cos t + \tfrac{1}{3} p_0^3 \sin^2 1'' \cos t \sin^2 t$$
$$\beta = \tfrac{1}{2} p_0^2 \sin 1'' \sin^2 t$$
$$\lambda = \tfrac{1}{3} p (p^2 - p_0^2) \sin^2 1'' \cos t \sin^2 t$$
$$\mu = \tfrac{1}{8} p^4 \sin^3 1'' \sin^4 t \tan^3 h$$

in which $p_0 = 1° \ 30' = 5400''$. Then, putting

$$A = \frac{p}{p_0}$$

$$\log A = \log p - 3.7323938$$

the formula (301) becomes

$$\varphi = h - (A\alpha + \lambda) + A^2 \beta \tan h + \mu$$

Putting then

$$x = A\alpha + \lambda$$
$$y = A^2\beta \tan h + \mu$$

we have

$$\varphi = h - x + y$$

$\left.\begin{array}{c}\\\\\\\\\end{array}\right\}$ (302)

or, when the zenith distance ζ is observed,

$$x = A\alpha + \lambda$$
$$y = A^2\beta \cot \zeta + \mu$$
$$90° - \varphi = \zeta + x - y$$

$\left.\begin{array}{c}\\\\\\\end{array}\right\}$ (303)

The first table gives α with the argument t; the second, β with the argument t; the third, λ with the arguments p and t; and the fourth, μ with the arguments y and φ, φ being used for h in so small a term.

To reduce a series of altitudes or zenith distances by these tables, we take for h or ζ the mean of the true altitudes or zenith distances; for α and β the means of the tabular numbers corresponding to the several hour angles, with which we find $A\alpha$ and $A^2\beta \cot \zeta$. The mean values of the very small quantities λ and μ are sensibly the same as the values corresponding to the mean of the hour angles; so that λ is taken out but once for the arguments polar distance and mean hour angle, and μ is taken with the arguments φ and the approximate value of $y = A^2\beta \cot \zeta$. Illustrative examples are given in connection with the tables.

FIFTH METHOD.—BY TWO ALTITUDES OF THE SAME STAR, OR DIF-
FERENT STARS, AND THE ELAPSED TIME BETWEEN THE OBSERVA-
TIONS.

178. Let S and S', Fig. 25, be any two points of
the celestial sphere, Z the zenith of the observer,
P the pole. Suppose that the altitudes of stars seen
at S and S', either at the same time or different
times, are observed, and that the observer has the
means of determining the angle SPS'; also that
the right ascensions and declinations of the stars
are known. From these data we can find *both the latitude and the
local time.* A graphic solution (upon an artificial globe) is indeed
quite simple, and it will throw light upon the analytic solution.
With the known polar distances of the stars and the angle SPS',

Fig. 25.
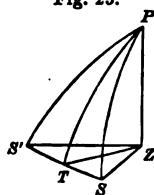
from the data h, h', δ, δ', and λ. I shall give two general solutions, the first of which is the one commonly known.

First Solution.—Let the observed points S and S' be joined by an arc of a great circle SS'. In the triangle PSS' there are given the sides $PS = 90° - \delta$, $PS' = 90° - \delta'$, and the angle $SPS' = \lambda$, from which we find the third side $SS' = B$, and the angle $PS'S = P$, by the formulæ [𝔄 of Art. 10]

$$\cos B = \sin \delta' \sin \delta + \cos \delta' \cos \delta \cos \lambda$$
$$\sin B \cos P = \cos \delta' \sin \delta - \sin \delta' \cos \delta \cos \lambda$$
$$\sin B \sin P = \cos \delta \ \sin \lambda$$

or, adapted for logarithmic computation,

$$\left.\begin{array}{r}
m \sin M = \sin \delta \\
m \cos M = \cos \delta \cos \lambda \\
\cos B = m \cos (M - \delta') \\
\sin B \cos P = m \sin (M - \delta') \\
\sin B \sin P = \cos \delta \sin \lambda
\end{array}\right\} \quad (305)$$

In the triangle ZSS' there are now known the three sides $ZS = 90° - h$, $ZS' = 90° - h'$, $SS' = B$, and hence the angle $ZS'S = Q$, by the formula employed in Art. 22:

$$\sin \tfrac{1}{2} Q = \sqrt{\left(\frac{\cos \tfrac{1}{2} (h' + h + B) \sin \tfrac{1}{2} (h' - h + B)}{\cos h' \sin B} \right)} \quad (306)$$

Now, putting
$$q = P - Q$$

there are known in the triangle PZS' the sides $PS' = 90° - \delta'$, $ZS' = 90° - h'$, and the angle $PS'Z = q$, from which the side $PZ = 90° - \varphi$, and the angle $S'PZ = t'$, are found by the formulæ

$$\sin \varphi = \sin \delta' \sin h' + \cos \delta' \cos h' \cos q$$
$$\cos \varphi \cos t' = \cos \delta' \sin h' - \sin \delta' \cos h' \cos q$$
$$\cos \varphi \sin t' = \cos h' \sin q$$

or, adapted for logarithmic computation,

$$\left.\begin{array}{r}
n \sin N = \sin h' \\
n \cos N = \cos h' \cos q \\
\sin \varphi = n \cos (N - \delta') \\
\cos \varphi \cos t' = n \sin (N - \delta') \\
\cos \varphi \sin t' = \cos h' \sin q
\end{array}\right\} \quad (307)$$

or, adapted for logarithmic computation, by introducing an auxiliary angle E,

$$\left. \begin{aligned} \sin C \sin E &= \sin \tfrac{1}{2} (\delta - \delta') \cos \tfrac{1}{2} \lambda \\ \sin C \cos E &= \cos \tfrac{1}{2} (\delta + \delta') \sin \tfrac{1}{2} \lambda \end{aligned} \right\} \quad (308)$$

In the triangle SPT we have the angle $PTS = P$, and therefore in the triangle $S'PT$ we have the angle $PTS' = 180° - P$, the cosine of which will be $= -\cos P$: hence, from these triangles we have the equations

$$\sin D \cos C + \cos D \sin C \cos P = \sin \delta$$
$$\sin D \cos C - \cos D \sin C \cos P = \sin \delta'$$

whence

$$2 \sin D \cos C = \sin \delta + \sin \delta'$$
$$2 \cos D \sin C \cos P = \sin \delta - \sin \delta'$$

$$\left. \begin{aligned} \sin D &= \frac{\sin \tfrac{1}{2} (\delta + \delta') \cos \tfrac{1}{2} (\delta - \delta')}{\cos C} \\[2mm] \cos P &= \frac{\cos \tfrac{1}{2} (\delta + \delta') \sin \tfrac{1}{2} (\delta - \delta')}{\cos D \sin C} \end{aligned} \right\} \quad (309)$$

which determine D and P after C has been found from (308).

In precisely the same manner we derive from the triangles ZTS and ZTS' the equations

$$\left. \begin{aligned} \sin H &= \frac{\sin \tfrac{1}{2} (h + h') \cos \tfrac{1}{2} (h - h')}{\cos C} \\[2mm] \cos Q &= \frac{\cos \tfrac{1}{2} (h + h') \sin \tfrac{1}{2} (h - h')}{\cos H \sin C} \end{aligned} \right\} \quad (310)$$

Then in the triangle PTZ we have the angle PTZ, by the formula

$$q = P - Q$$

and hence the equations

$$\sin \varphi = \sin D \sin H + \cos D \cos H \cos q$$
$$\cos \varphi \cos \tau = \cos D \sin H - \sin D \cos H \cos q$$
$$\cos \varphi \sin \tau = \qquad\qquad\qquad \cos H \sin q$$

To adapt these for logarithmic computation, let β and γ be determined by the conditions*

$$\left.\begin{array}{r} \cos \beta \sin \gamma = \cos H \cos q \\ \cos \beta \cos \gamma = \sin H \\ \sin \beta = \cos H \sin q \end{array}\right\} \quad (311)$$

then φ and τ are found by the equations

$$\left.\begin{array}{r} \sin \varphi = \cos \beta \sin (D + \gamma) \\ \cos \varphi \cos \tau = \cos \beta \cos (D + \gamma) \\ \cos \varphi \sin \tau = \sin \beta \end{array}\right\} \quad (312)$$

To find the hour angles t and t', let

$$x = \tau - \tfrac{1}{2}(t' + t)$$

then, since $\tfrac{1}{2}\lambda = \tfrac{1}{2}(t' - t)$, we have

$$\begin{array}{l} \tfrac{1}{2}\lambda + x = \tau - t = \text{the angle } TPS, \\ \tfrac{1}{2}\lambda - x = t' - \tau = \text{the angle } TPS', \end{array}$$

and from the triangles PTS and PTS' we have

$$\frac{\sin(\tfrac{1}{2}\lambda + x)}{\sin C} = \frac{\sin P}{\cos \delta} \qquad \frac{\sin(\tfrac{1}{2}\lambda - x)}{\sin C} = \frac{\sin P}{\cos \delta'}$$

whence

$$\frac{\sin(\tfrac{1}{2}\lambda + x) - \sin(\tfrac{1}{2}\lambda - x)}{\sin(\tfrac{1}{2}\lambda + x) + \sin(\tfrac{1}{2}\lambda - x)} = \frac{\cos \delta' - \cos \delta}{\cos \delta' + \cos \delta}$$

and, consequently,

$$\tan x = \tan \tfrac{1}{2}(\delta + \delta') \tan \tfrac{1}{2}(\delta - \delta') \tan \tfrac{1}{2}\lambda \quad (313)$$

Hence, finally,

$$\left.\begin{array}{l} t = \tau - x - \tfrac{1}{2}\lambda \\ t' = \tau - x + \tfrac{1}{2}\lambda \end{array}\right\} \quad (314)$$

As in the first solution, the value of q will become $= P + Q$ when the arc joining the observed places of the stars passes north of the zenith.

EXAMPLE.—1856 March 5, in the assumed Latitude 39° 17′ N. and Longitude 5ʰ 6ᵐ 36ˢ W., suppose the following altitudes

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* The equations (311) can always be satisfied, since the sum of their squares gives the identical equation $1 = 1$.

(The auxiliaries C and H are not themselves required: we take their cosines from the table, employing the sines as arguments.) We now find, by (311), (312), (313), and (314),

$$\log \sin \beta = n9.673029 \qquad \tau = 322°\ 30'\ 51''.3$$
$$\log \cos \beta = \ \ 9.945532 \qquad x = \ \ \ \ 1\ \ 14\ \ 21\ .3$$
$$\gamma = 39°\ 18'\ \ 4''.0 \qquad T - x = 321\ \ 16\ \ 30$$
$$D + \gamma = 45\ \ 52\ \ 36\ .0 \qquad = 21^{\text{h}}\ 25^{\text{m}}\ \ 6^{\text{s}}$$
$$\varphi = 39\ \ 17\ \ 20\ . \qquad \tfrac{1}{2}\lambda = \ \ \ 2\ \ 55\ \ \ 6$$
$$t = \ 18\ \ 30\ \ \ 0$$
$$t' = \ \ \ 0\ \ 20\ \ 12$$

agreeing precisely with the results of the first solution.

179. In the observation of lunar distances, as we shall see hereafter, the altitudes of the moon and a star are observed at the same instant with the distance of the objects. The observed distance is reduced to the true geocentric distance, which is the arc B of the above *first solution*, or $2\,C$ of the *second*. The observation of a lunar distance with the altitudes of the objects furnishes, therefore, the data for finding the latitude, the local time, and the longitude.

180. (B.) *A fixed star observed at two different times.*—In this case the declination is the same at both observations, and the general formulæ of the preceding articles take much more simple forms. The hour angle λ is in this case merely the elapsed sidereal time between the observations, the formula (304), since $a = a'$, becoming here

$$\lambda = (T' - T) + (\Delta T' - \Delta T) \tag{315}$$

Putting δ' for δ in (308) and (309), we find $E = 0$, $\cos P = 0$, $P = 90°$; and C and D are found by the equations

$$\sin C = \cos \delta \sin \tfrac{1}{2}\lambda, \qquad \sin D = \frac{\sin \delta}{\cos C} \tag{316}$$

Since we have $q = P - Q = 90° - Q$, the last two equations of (311) give

$$\sin \beta = \cos H \cos Q, \qquad \cos \gamma = \sin H \sec \beta$$

which, by (310), become*

$$\sin \beta = \frac{\cos \frac{1}{2}(h + h') \sin \frac{1}{2}(h - h')}{\sin C}$$

$$\cos \gamma = \frac{\sin \frac{1}{2}(h + h') \cos \frac{1}{2}(h - h')}{\cos \beta \cos C}$$
$$\left.\right\} \quad (317)$$

Then φ and τ are found by (312), or rather by the following:

$$\sin \varphi = \cos \beta \sin (D + \gamma)$$

$$\tan \tau = \frac{\tan \beta}{\cos(D + \gamma)} \quad \text{or} \quad \sin \tau = \frac{\sin \beta}{\cos \varphi}$$
$$\left.\right\} \quad (318)$$

The hour angles at the two observations are

$$\begin{aligned} t &= \tau - \tfrac{1}{2}\lambda \\ t' &= \tau + \tfrac{1}{2}\lambda \end{aligned} \quad \left.\right\} \quad (319)$$

Here γ, being determined by its cosine, may be either a positive or a negative angle, and we obtain two values of the latitude by taking either $D + \gamma$ or $D - \gamma$ in (318). The first value will be taken when the great circle joining the two positions of the star passes north of the zenith; the second, when it passes south of the zenith.

The solution may be slightly varied by finding D by the formula

$$\tan D = \frac{\tan \delta}{\cos \frac{1}{2}\lambda} \quad (320)$$

obtained directly from the triangle PTS (Fig. 25), which is right-angled at T when the declinations are equal. We can then dispense with C by writing the formulæ (317) as follows:

$$\sin \beta = \frac{\cos \frac{1}{2}(h + h') \sin \frac{1}{2}(h - h')}{\cos \delta \sin \frac{1}{2}\lambda}$$

$$\cos \gamma = \frac{\sin \frac{1}{2}(h + h') \cos \frac{1}{2}(h - h') \sin D}{\cos \beta \sin \delta}$$
$$\left.\right\} \quad (321)$$

---

* The formulæ (315), (316), and (317) are essentially the same as those first given for this case by M. CAILLET in his *Manuel du Navigateur*, Nantes, 1818.

181. (C.) *The sun observed at two different times.*—In this case the hour angle λ is the elapsed apparent solar time. If then the times T and T' are observed by a mean solar chronometer, and the equation of time at the two observations is e and e' (positive when additive to apparent time), we have

$$\lambda = (T' - T) + (\Delta T' - \Delta T) - (e' - e) \qquad (322)$$

Taking then the declinations δ and δ' for the two times of observation, we can proceed by the general methods of Art. 178.

But, as the declinations differ very little, we can assume their mean—*i.e.* the declination at the middle instant between the observations—as a constant declination, and compute at least an approximate value of the latitude by the simple formulæ for a fixed star in the preceding article. We can, however, readily correct the resulting latitude for the error of this assumption. To obtain the correction, we recur to the rigorous formulæ of our second solution in Art. 178. The change of the sun's declination being never greater than $1'$ per hour, we may put $\cos \frac{1}{2}(\delta - \delta') = 1$. Making this substitution in (308) and (309), and putting δ for $\frac{1}{2}(\delta + \delta')$ so that δ will signify the mean of the declinations, and $\Delta\delta$ for $\frac{1}{2}(\delta' - \delta)$ so that $\Delta\delta$ will signify one-half the *increase* of the sun's declination from the first to the second observation (positive when the sun is moving northward), we shall have

$$\Delta \delta = -\tfrac{1}{2}(\delta - \delta')$$

$$\tan E = -\frac{\sin \Delta \delta}{\cos \delta \tan \frac{1}{2} \lambda}$$

But $\Delta\delta$ diminishes with λ, so that E always remains a small quantity of the same order as $\Delta\delta$; and hence we may also put $\cos E = 1$. Thus the second equation of (308) gives

$$\sin C = \cos \delta \sin \tfrac{1}{2} \lambda$$

and the first of (309)

$$\sin D = \frac{\sin \delta}{\cos C}$$

which are the same as (316), as given for the case where the declination is absolutely invariable. Hence no sensible error is produced in the values of C and D by the use of the mean de-

clination. But by the second equation of (309) P will no longer be exactly 90°: if then we put

$$P = 90° + \Delta P$$

we have, by that equation,

$$\sin \Delta P = \frac{\cos \delta \sin \Delta \delta}{\cos D \sin C} = \frac{\sin \Delta \delta}{\cos D \sin \frac{1}{2} \lambda}$$

or simply

$$\Delta P = \frac{\Delta \delta}{\cos D \sin \frac{1}{2} \lambda}$$

Then, since $q = P - Q$, we have

$$q = 90° - Q + \Delta P$$

The rigorous formula for the latitude is

$$\sin \varphi = \sin D \sin H + \cos D \cos H \cos q$$

in which when we use the mean declination we take $q = 90° - Q$: therefore, to find the correction of φ corresponding to a correction of $q = \Delta P$, we have by differentiating this equation, D and H being invariable,

$$\cos \varphi . \Delta \varphi = - \cos D \cos H \sin q . \Delta P$$

$$= - \frac{\Delta \delta \cos H \cos Q}{\sin \frac{1}{2} \lambda}$$

We have found in the preceding article $\sin \beta = \cos H \cos Q$; and hence

$$\Delta \varphi = - \frac{\Delta \delta \sin \beta}{\cos \varphi \sin \frac{1}{2} \lambda} \tag{323}$$

In the case of the sun, therefore, we compute the approximate latitude φ by the formulæ (316), (317), and (318), employing for δ the mean declination. We then find $\Delta \varphi$ by (323) (which involves very little additional labor, since the logarithms of $\sin \beta$ and $\sin \frac{1}{2} \lambda$ have already occurred in the previous computation), and then we have the true latitude

$$\varphi' = \varphi + \Delta \varphi$$

If we wish also to correct the hour angle τ found by this method, we have, from the second equation of (47) applied to

the triangle PTZ (taking b and c to denote the sides PT and ZT, which are here constant),

$$\Delta\tau = \frac{\cos H \cos A}{\cos \varphi} \cdot \Delta P$$

in which A is the azimuth of the point T. But we have in the triangle PTZ

$$\cos H \cos A = \sin \varphi \cos D \cos \tau - \cos \varphi \sin D$$

Substituting this and the value of ΔP, we have

$$\Delta\tau = \frac{\Delta\delta}{\sin \frac{1}{2}\lambda} (\tan \varphi \cos \tau - \tan D)$$

and, substituting the value of $\tan D$ (320),

$$\Delta\tau = \frac{\Delta\delta}{\sin \frac{1}{2}\lambda} \left(\tan \varphi \cos \tau - \frac{\tan \delta}{\cos \frac{1}{2}\lambda} \right)$$

When this correction is added to τ, we have the value that would be found by the rigorous formulæ, and we have then to apply also the correction x according to (314). In the present case we have, by (313),

$$x = - \Delta\delta \tan \delta \tan \frac{1}{2}\lambda$$

and the complete formulæ for the hour angles t and t' become

$$t = \tau + \Delta\tau - x - \frac{1}{2}\lambda$$
$$t' = \tau + \Delta\tau - x + \frac{1}{2}\lambda$$

Putting

$$y = \Delta\tau - x$$

we find, by substituting the above values of $\Delta\tau$ and x,

$$y = \Delta\delta \cdot \left(\frac{\tan \varphi \cos \tau}{\sin \frac{1}{2}\lambda} - \frac{\tan \delta}{\tan \frac{1}{2}\lambda} \right) \qquad (324)$$

and then we have

$$t = \tau + y - \frac{1}{2}\lambda$$
$$t' = \tau + y + \frac{1}{2}\lambda \qquad \Bigg\} \quad (325)$$

The corrections $\Delta\varphi$ and y are computed with sufficient accuracy with four-place logarithms, and, therefore, add but little to the labor of the computation. Nevertheless, when both latitude and time are required with the greatest precision, it will be preferable to compute by the rigorous formulæ.

EXAMPLE.—1856 March 10, in Lat. 24° N., Long. 30° W., suppose we obtain two altitudes of the sun as follows, all corrections being applied: find the latitude.

At app. time $\quad$ 0ʰ 30ᵐ $\qquad$ $h = 61° 11' 38''.3 \quad (\delta) = -3° 51' 52''.8$

" $\qquad$ 4 30 $\qquad$ $h' = 18 \quad 46 \quad 35 .8 \quad (\delta') = -3 \quad 47 \quad 57 .4$

$\tfrac{1}{2}\lambda = \quad$ 2ʰ 0ᵐ $\quad \tfrac{1}{2}(h+h') = 39 \quad 59 \quad 7 .1 \quad \delta = -3 \quad 49 \quad 55 .1$

$= 30° .0' \quad \tfrac{1}{2}(h-h') = 21 \quad 12 \quad 31 .3 \quad \Delta\delta = \quad + \quad 1' 57''.7$

The following is the form of computation by the formulæ (316), (317), and (318), employed by BOWDITCH in his *Practical Navigator*, the reciprocals of the equations (316) and of the second of (317) being used to avoid taking arithmetical complements. This form is convenient when the tables give the secants and cosecants, as is usual in nautical works.

cosec $\tfrac{1}{2}\lambda$ $\quad$ 0.301080

sec δ $\quad$ 0.000972 $\quad$ . . . . $\quad$ cosec n1.175024

cosec C $\quad$ 0.302002 $\quad$ cos $\;$ 9.937854 $\quad$ . . . . $\quad$ cos $\quad$ 9.937854

cos $\tfrac{1}{2}(h+h')$ 9.884347 $\quad$ cosec 0.192065 $\quad D = -4° 25' 21''.8 \quad$ cosec n1.112878

sin $\tfrac{1}{2}(h-h')$ 9.558428 $\quad$ sec $\quad$ 0.030459

sin β $\quad$ 9.744777 $\quad$ cos $\quad$ 9.919829 $\quad$ . . . . $\quad$ cos $\quad$ 9.919829

$\qquad\qquad\qquad\quad$ sec $\quad$ 0.080207 $\quad \gamma = \quad$ 33 $\quad$ 45 $\quad$ 38 .0 $\quad$ sec

$\qquad\qquad\qquad\qquad\qquad\qquad D + \gamma = \quad$ 29 $\quad$ 20 $\quad$ 16 .7 $\quad$ sin $\quad$ 9.690161

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \phi = \quad$ 24° $\quad$ 2' $\quad$ 23''.2 $\quad$ sin $\quad$ 9.609990

If the approximate latitude had not been given, we might also have taken $D - \gamma = -33° 45' 38''.0$, and then we should have had

$$\cos \beta \quad 9.919829$$
$$\sin (D - \gamma) \quad n9.791113$$
$$\varphi = -30° 55' 44''.3 \qquad \sin \varphi \quad n9.710942$$

To correct the first value of the latitude for the change of declination, we have, by (323),

$$\log \Delta\delta \quad 2.0708$$
$$\sin \beta \quad 9.7448$$
$$\text{cosec } \tfrac{1}{2}\lambda \quad 0.3010$$
$$\sec \varphi \quad 0.0394$$
$$\Delta\varphi = -143''.2 \quad \log \Delta\varphi \quad n2.1560$$

and hence the true latitude is

$$\varphi' = 24° 2' 23''.2 - 2' 23''.2 = 24° 0' 0''$$

which agrees exactly with the value computed by the rigorous formulæ.

The approximate time is found by the last equation of (318) with but one new logarithm: we have

$$\begin{array}{ll} & \sin \beta \; 9.744777 \\ & \cos \varphi \; 9.960596 \\ \tau = 37° \; 28' \; 23'' & \sin \tau \; 9.784181 \end{array}$$

By (324) and (325), we find

$$\begin{array}{llll} \log \Delta\delta & 2.0708 & \log \Delta\delta & 2.0708 \\ \operatorname{cosec} \frac{1}{2}\lambda & 0.3010 & \cot \frac{1}{2}\lambda & 0.2386 \\ \tan \varphi & 9.6494 & \tan \delta & n8.8259 \\ \cos \tau & 9.8996 & -13''.7 & n1.1353 \\ +83''.3 & 1.9208 \end{array}$$

$$y = + 83''.3 - (-13''.7) = + 97''$$
$$\tau + y = 37° \; 30' \; 0'' = 2^h \; 30^m \; 0^s$$
$$t = \; 0^h \; 30^m \; 0^s \qquad t' = 4^h \; 30^m \; 0^s$$

which are perfectly exact.

182. (D.) *Two equal altitudes of the same star, or of the sun.*—This case is a very useful one in practice with the sextant when equal altitudes have been taken for determining the time by the method of Art. 140. By putting $h' = h$ in the formulæ (317), we find $\sin \beta = 0$, $\cos \beta = 1$, and hence (318) gives $\sin \varphi = \sin (D + \gamma)$, or $\varphi = D + \gamma$. We have, therefore, for this case, by (320) and (321),

$$\left. \tan D = \frac{\tan \delta}{\cos \frac{1}{2}\lambda} \qquad \cos \gamma = \frac{\sin h \sin D}{\sin \delta} \atop \varphi = D \pm \gamma \right\} \quad (326)$$

which is the method of Art. 164 applied as proposed in Art. 165. We give γ the double sign, and obtain two values of the latitude, for the reasons already stated.

The time will be most conveniently found by Art. 140. The method there given is, however, embraced in the solution of the present problem. For, since we here have $\sin \beta = 0$, we also have $\tau = 0$, and the hour angles given by (325) become

$$t = y - \frac{1}{2}\lambda$$
$$t' = y + \frac{1}{2}\lambda$$

(328), $d\varphi = + dh$, and the total correction of the latitude $= dh - d\delta$, precisely the same as if the meridian observation were the only one. Hence there is no advantage in combining an observation on the meridian with another remote from it, in the determination of latitude: a simple meridian observation, or, still better, a series of circummeridian observations, is always preferable if the time is approximately known.

. When the sun is observed and the mean declination is employed, putting $\Delta\delta = \frac{1}{2}(\delta' - \delta)$, we have $d\delta = \Delta\delta$, $d\delta' = -\Delta\delta$, and, by (332),

$$\Delta\varphi = -\frac{\sin A' \cos q + \sin A \cos q'}{\sin (A' - A)} \cdot \Delta\delta$$

which, by substituting

$$\sin A' = \frac{\sin q \cos \delta}{\cos \varphi} \qquad \sin A = \frac{\sin q' \cos \delta}{\cos \varphi}$$

becomes

$$\Delta\varphi = -\frac{\sin (q' + q) \cos \delta}{\sin (A' - A) \cos \varphi} \cdot \Delta\delta \qquad (333)$$

This is but another expression of the correction (323).

If, when the sun is observed, instead of employing the mean declination we employ the declination belonging to the *greater* altitude, which we may suppose to be h, we shall have $d\delta = 0$, $d\delta' = -2\Delta\delta$; and, denoting the correction of the latitude in this case by $\Delta'\varphi$, we have, by (332),

$$\Delta'\varphi = -\frac{2 \sin A \cos q'}{\sin (A' - A)} \cdot \Delta\delta = -\frac{2 \sin q \cos q' \cos \delta}{\sin (A' - A) \cos \varphi} \cdot \Delta\delta$$

Under what conditions will $\Delta'\varphi$ be numerically less than $\Delta\varphi$?

First. If both observations are on the same side of the meridian, the condition $\Delta'\varphi < \Delta\varphi$ gives

$$2 \sin q \cos q' < \sin (q' + q)$$

or

$$2 \sin q \cos q' < \sin q' \cos q + \cos q' \sin q$$

whence

$$\tan q < \tan q'$$

which condition is always satisfied when h is the greater altitude.

Secondly. If the observations are on different sides of the

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meridian, q and q' will have opposite signs, and we shall have, numerically, $\sin (q' - q)$ instead of $\sin (q' + q)$. The condition $\Delta'\varphi < \Delta\varphi$, then, requires that

$$2 \sin q \cos q' < \sin q' \cos q - \cos q' \sin q$$

or

$$\tan q < \tfrac{1}{3} \tan q'$$

Therefore $\Delta \varphi$ will be greater than $\Delta\varphi$ *only* when the observations are on opposite sides of the meridian and $\tan q > \tfrac{1}{3} \tan q'$. In cases where an approximate result suffices, as at sea, and the correction $\Delta\varphi$ is omitted to save computation, it will be expedient to employ the declination at the greater altitude, except in the single case just mentioned.* But to distinguish this case with accuracy we should have to compute the angles q and q'; and therefore an approximate criterion must suffice. Since the parallactic angles increase with the hour angles, we may substitute for the condition $\tan q > \tfrac{1}{3} \tan q'$ the more simple one $t > \tfrac{1}{3} t'$, which gives

$$t > \frac{t' - t}{2}$$

or (t and t' being only the numerical values of the hour angles)

$$t > \tau$$

Hence we derive this very simple practical rule: *employ the sun's declination at the greater altitude, except when the time of this altitude is farther from noon than the middle time, in which case employ the mean declination.*

The greatest error committed under this rule is (nearly) the value of $\Delta\varphi$ in (323), when $\tau = t$. But since in this case $3t = t'$, and $t + t' = \lambda$, we have $\tau = \tfrac{1}{4} \lambda$, and therefore $\sin \beta = \cos \varphi \sin \tau = \cos \varphi \sin \tfrac{1}{4} \lambda$. This reduces (323) to $\Delta\varphi = - \tfrac{1}{2} \Delta\delta \sec \tfrac{1}{4} \lambda$. Since λ will seldom exceed 6^{h}, $\Delta\delta$ will not exceed $3'$, and the maximum error will not exceed $1'.6$. In most cases the error will be under $1'$, a degree of approximation usually quite sufficient at sea. Nevertheless, the computation of the correction $\Delta\varphi$ by our formula (323) is so simple that the careful navigator

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* Bowditch and navigators generally employ in all cases the mean declination; but the above discussion proves that, if the cases are not to be distinguished, it will be better always to employ the declination at the greater altitude.

will prefer to employ the mean declination and to obtain the exact result by applying this correction in all cases.

SIXTH METHOD.—BY TWO ALTITUDES OF THE SAME OR DIFFERENT STARS, WITH THE DIFFERENCE OF THEIR AZIMUTHS.

184. Instead of noting the times corresponding to the observed altitudes, we may observe the azimuths, both altitude and azimuth being obtained at once by the Altitude and Azimuth Instrument or the Universal Instrument. The instrument gives the *difference* of azimuths $= \lambda$. The computation of the latitude and the absolute azimuth A of one of the stars may then be performed by the formulæ of the preceding method, only interchanging altitudes and declinations and putting $180° - A$ for t. When A has been found, we may also find t by the usual methods.

SEVENTH METHOD.—BY TWO DIFFERENT STARS OBSERVED AT THE SAME ALTITUDE WHEN THE TIME IS GIVEN.

185. By this method the latitude is found from the declinations of the two stars and their hour angles deduced from the times of observation, *without employing the altitude itself*, so that the result is free from constant errors (of graduation, &c.) of the instrument with which the altitude is observed. Let

$$\Theta, \Theta' = \text{the sidereal times of the observations,}$$
$$\alpha, \alpha' = \text{the right ascensions of the stars,}$$
$$\delta, \delta' = \text{the declinations} \quad \text{``} \quad \text{``}$$
$$t, t' = \text{the hour angles} \quad \text{``} \quad \text{``}$$
$$h = \text{the altitude of either star,}$$
$$\varphi = \text{the latitude;}$$

then, the hour angles being found by the equations

$$t = \Theta - \alpha \qquad t' = \Theta' - \alpha'$$

we have

$$\sin h = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t$$
$$\sin h = \sin \varphi \sin \delta' + \cos \varphi \cos \delta' \cos t'$$

From the difference of these we deduce

$$\tan \varphi (\sin \delta' - \sin \delta) = \cos \delta \cos t - \cos \delta' \cos t'$$
$$= \tfrac{1}{2} (\cos \delta - \cos \delta') (\cos t + \cos t')$$
$$+ \tfrac{1}{2} (\cos \delta + \cos \delta') (\cos t - \cos t')$$

and, by resolving the quantities in parentheses into their factors,

$$\tan \varphi = \tan \tfrac{1}{2} (\delta' + \delta) \cos \tfrac{1}{2} (t' + t) \cos \tfrac{1}{2} (t' - t)$$
$$+ \cot \tfrac{1}{2} (\delta' - \delta) \sin \tfrac{1}{2} (t' + t) \sin \tfrac{1}{2} (t' - t)$$

If now we put

$$\left.\begin{array}{l} m \sin M = \sin \tfrac{1}{2} (t' - t) \cot \tfrac{1}{2} (\delta' - \delta) \\ m \cos M = \cos \tfrac{1}{2} (t' - t) \tan \tfrac{1}{2} (\delta' + \delta) \end{array}\right\} \text{(334)}$$

we have

$$\tan \varphi = m \cos [\tfrac{1}{2} (t' + t) - M] \qquad (335)$$

The equations (334) determine m and M, and then the latitude is found by (335) in a very simple manner.

It is important to determine the conditions which must govern the selection of the stars and the time at which they are to be observed. For this purpose we differentiate the above expressions for sin h relatively to φ and t. The error in the hour angles is composed of the error of observation and the error of the clock correction. If we put

$$T, T' = \text{the observed (sidereal) clock time,}$$
$$\Delta T = \text{the clock correction,}$$
$$\delta T = \text{the rate of the clock in a unit of clock time,}$$

we shall have

$$t = T + \Delta T - \alpha, \quad t' = T' + \Delta T + \delta T(T' - T) - \alpha'$$

Differentiating these, assuming that the *rate* of the clock is sufficiently well known, we have

$$dt = dT + d\Delta T, \quad dt' = dT' + d\Delta T$$

in which dT, dT' are the errors in the observed times, and $d\Delta T$ the error in the clock correction. The differential equations are then

$$dh = - \cos A \, d\varphi - \cos \varphi \sin A \, dT - \cos \varphi \sin A \, d\Delta T$$
$$dh = - \cos A' \, d\varphi - \cos \varphi \sin A' \, dT' - \cos \varphi \sin A' \, d\Delta T$$

in which A and A' are the azimuths of the stars. The difference of these equations gives

$$\frac{d\varphi}{\cos \varphi} = - \frac{\sin A}{\cos A - \cos A'} dT + \frac{\sin A'}{\cos A - \cos A'} dT' + \frac{\sin A' - \sin A}{\cos A' - \cos A} d\Delta T$$

The denominator $\cos A - \cos A'$ is a maximum when one of the azimuths is zero and the other 180°, that is, when one of the stars is south and the other north of the observer. To satisfy this condition as nearly as possible, two stars are to be selected the mean of whose declinations is nearly equal to the latitude, and the common altitude at which they are to be observed will be equal to or but little less than the meridian altitude of the star which culminates farthest from the zenith. It is desirable, also, that the difference of right ascensions should not be great.

The coefficient of $d\Delta T$ is equal to $-\cot \frac{1}{2}(A' + A)$, which is zero when $\frac{1}{2}(A' + A)$ is 90° or 270°: hence, when the observations are equally distant from the prime vertical, one north and the other south, small errors in the clock correction have no sensible effect.

When the latitude has been found by this method, we may determine the whole error of the instrument with which the altitude is observed; for either of the fundamental equations will give the true altitude, which increased by the refraction will be that which a perfect instrument would give.

186. With the zenith telescope (see Vol. II.) the two stars may be observed at *nearly* the same zenith distance, the small difference of zenith distance being determined by the level and the micrometer. The preceding method may still be used by correcting the time of one of the observations. If at the observed times T, T' the zenith distances are ζ and ζ', the times when the same altitudes would be observed are either

$$T \quad \text{and} \quad T' + \frac{\zeta - \zeta'}{\cos \varphi \sin A'}$$

or,

$$T + \frac{\zeta' - \zeta}{\cos \varphi \sin A} \quad \text{and} \quad T'$$

where $\zeta' - \zeta$ is given directly by the instrument. With the hour angles deduced from these times we may then proceed by (334) and (335).

But it will be still better to compute an approximate latitude by the formulæ (334) and (335), employing the actually observed times, and then to correct this latitude for the difference of zenith distance.

By differentiating the formula

$$\tan \varphi (\sin \delta' - \sin \delta) = \cos \delta \cos t - \cos \delta' \cos t'$$

relatively to φ and t', we have

$$\sec^2 \varphi (\sin \delta' - \sin \delta) \, d\varphi = \cos \delta' \sin t' \, dt' = \sin \zeta \sin A' \, dt'$$

Hence, substituting

$$dt' = dT' = \frac{\zeta - \zeta'}{\cos \varphi \sin A'},$$

we find

$$d\varphi = \frac{\frac{1}{2}(\zeta - \zeta') \sin \zeta \cos \varphi}{\sin \frac{1}{2}(\delta' - \delta) \cos \frac{1}{2}(\delta' + \delta)} \qquad (336)$$

and the true latitude will be $\varphi + d\varphi$.

EIGHTH METHOD.—BY THREE OR MORE DIFFERENT STARS OBSERVED AT THE SAME ALTITUDE WHEN THE TIME IS NOT GIVEN.

187. *To find both the latitude and the clock correction from the clock times when three different stars arrive at the same altitude.*

As in the preceding method, we do not employ the common altitude itself; and, as we have one more observation, we can determine the time as well as the latitude.

Let S, S', S'', Fig. 26, be the three points of the celestial sphere, equidistant from the zenith Z, at which the stars are observed.   Let
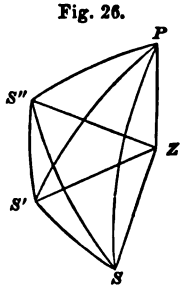
Fig. 26.

T, T', T'' = the clock times of the observations,
ΔT = the clock correction to sidereal time at the time T,
δT = the rate of the clock in a unit of clock time,
a, a', a'' = the right ascensions of the stars,
δ, δ', δ'' = the declinations   "   "
t, t', t'' = the hour angles   "   "
h = the altitude,
φ = the latitude.

Also, let

$$\lambda = t' - t = SPS',$$
$$\lambda' = t'' - t = SPS'';$$

then, since the sidereal times of the observations are

there follows also

$$k \cos (t + N') = \frac{1}{m'}$$

and these two equations are of the form treated of in **Pl. Trig.** Art. 179, according to which, if the auxiliary ϑ is determined by the condition

$$\tan \vartheta = \frac{m}{m'}, \qquad (343)$$

we shall have

$$\tan [t + \tfrac{1}{2} (N + N')] = \tan (45° - \vartheta) \cot \tfrac{1}{2} (N' - N) \qquad (344)$$

which determines t, from which the clock correction is found by the formula

$$\Delta T = a + t - T$$

The latitude is then found from either (339) or (342).[*]

To determine the conditions which shall govern the selection of the stars, we have, as in Art. 185,

$$dh = - \cos A \; d\varphi - \cos \varphi \sin A \; dT \; - \cos \varphi \sin A \; d\Delta T$$
$$dh = - \cos A' \; d\varphi - \cos \varphi \sin A' \; dT' - \cos \varphi \sin A' \; d\Delta T$$
$$dh = - \cos A'' d\varphi - \cos \varphi \sin A'' dT'' - \cos \varphi \sin A'' d\Delta T$$

By the elimination of $d\Delta T$, we deduce the following:

$$(\sin A \; - \sin A' \;) \, dh = \sin (A' - A \;) \, d\phi - \cos \phi \sin A' \sin A \;\; (dT'' \; - dT \;)$$
$$(\sin A' - \sin A'') \, dh = \sin (A'' - A') \, d\phi - \cos \phi \sin A'' \sin A' \, (dT'' - dT' \,)$$
$$(\sin A'' - \sin A \;) \, dh = \sin (A \; - A'') \, d\phi - \cos \phi \sin A \;\, \sin A''(dT \; - dT'')$$

Adding these three equations together, and putting

$$2K = \sin (A' - A) + \sin (A'' - A') + \sin (A - A'')$$

we find

$$\frac{d\phi}{\cos \phi} = \frac{\sin A (\sin A'' - \sin A')}{2K} \, dT + \frac{\sin A' (\sin A - \sin A'')}{2K} \, dT''$$

$$+ \frac{\sin A'' (\sin A' - \sin A)}{2K} \, dT''$$

By eliminating $d\varphi$ from the same three equations, we shall find

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[*] This simple and elegant solution is due to GAUSS, *Monatliche Correspondenz*, Vol. XVIII. p. 287.

$$\alpha \; Andromedæ \quad T = 21^h \; 33^m \; 26^s$$
$$\alpha \; Ursæ \; Minoris \; T' = 21 \quad 47 \quad 30$$
$$\alpha \; Lyræ \qquad T'' = 22 \quad 5 \quad 21$$

The apparent places of the stars were as follows:

$$\alpha \; Andromedæ \quad \alpha = 23^h \; 58^m \; 33^s.33 \qquad \delta = 28° \; 2' \; 14''.8$$
$$\alpha \; Ursæ \; Minoris \; \alpha' = \; 0 \quad 55 \quad 4.70 \qquad \delta' = 88 \quad 17 \quad 5\;.7$$
$$\alpha \; Lyræ \qquad \alpha'' = 18 \quad 30 \quad 28.96 \qquad \delta'' = 38 \quad 37 \quad 6\;.6$$

Hence we find

$\frac{1}{2}\lambda = - \quad 5° \; 18' \; 25''.28$	$\frac{1}{2}\lambda' = 44° \; 59' \; 55''.28$
$\frac{1}{2}(\delta' - \delta) = \quad 30 \quad 7 \; 25 \; .45$	$\frac{1}{2}(\delta'' - \delta) = \; 5 \quad 17 \; 25 \; .90$
$\frac{1}{2}(\delta' + \delta) = \quad 58 \quad 9 \; 40 \; .25$	$\frac{1}{2}(\delta'' + \delta) = 33 \quad 19 \; 40 \; .70$

$\log \cot \frac{1}{2}(\delta' - \delta) \quad 0.2363973$	$\log \cot \frac{1}{2}(\delta'' - \delta) \quad 1.0333869$
$\log \sin \frac{1}{2}\lambda \qquad n8.9661070$	$\log \sin \frac{1}{2}\lambda' \qquad 9.8494751$
$\log m \sin M \qquad n9.2025043$	$\log m' \sin M' \qquad 0.8828620$

$\log \tan \frac{1}{2}(\delta' + \delta) \quad 0.2069331$	$\log \tan \frac{1}{2}(\delta'' + \delta) \; 9.8179461$
$\log \cos \frac{1}{2}\lambda \qquad 9.9981343$	$\log \cos \frac{1}{2}\lambda' \qquad 9.8494949$
$\log m \cos M \qquad 0.2050674$	$\log m' \cos M' \qquad 9.6674410$

$\log \tan M \qquad n8.9974369$	$\log \tan M' \qquad 1.2154210$
$\log \cos M \qquad 9.9978645$	$\log \sin M' \qquad 9.9991963$
$\log m \qquad 0.2072029$	$\log m' \qquad 0.8836657$

$$M = - 5° \; 40' \; 37''.96 \qquad M' = \quad 86° \; 30' \; 55''.07$$
$$\frac{1}{2}\lambda - M = N = + 0 \quad 22 \quad 12 \; .68 \qquad \frac{1}{2}\lambda' - M' = N' = - 41 \quad 30 \quad 59 \; .79$$

$$\vartheta = \quad 11° \; 53' \; 41''.28 \; \log \frac{m}{m'} = \log \tan \vartheta \qquad 9.3235372$$
$$45° - \vartheta = \quad 33 \quad 6 \; 18 \; .72 \; \log \tan (45° - \vartheta) \qquad 9.8142617$$
$$\frac{1}{2}(N' - N) = - 20 \quad 56 \; 36 \; .24 \; \log \cot \frac{1}{2}(N' - N) \qquad n0.4171063$$
$$t + \frac{1}{2}(N' + N) = - 59 \quad 35 \; 14 \; .71 \; \log \tan [t + \frac{1}{2}(N' + N)] \; n0.2313680$$
$$\frac{1}{2}(N' + N) = - 20 \quad 34 \; 23 \; .56$$
$$t = - 39 \quad 0 \; 51 \; .15 = - \quad 2^h \; 36^m \; 3^s.41$$
$$\alpha = \quad 23 \quad 58 \quad 33 \; .33$$
$$t + \alpha = \Theta = \quad 21 \quad 22 \quad 29 \; .92$$
$$T = \quad 21 \quad 33 \quad 26 \; .$$
$$\text{Clock correction } \Delta T = \quad - \; 10 \quad 56 \; .08$$

Then, to find the latitude, we have

$$t + N = -38° \ 38' \ 38''.47 \qquad t + N' = -80° \ 31' \ 50''.94$$

$\log \cos (t + N)$	9.8926738	$\log \cos (t + N')$	9.2162110
$\log m$	0.2072029	$\log m'$	0.8836657
$\log \tan \varphi$	0.0998767	$\log \tan \varphi$	0.0998767

$$\varphi = 51° \ 31' \ 51''.46$$

If with these results we compute the true altitude of the stars, we find from each $h = 52° \ 37' \ 21''.2$. The refraction was $42''.7$, and hence the apparent altitude $= 52° \ 38' \ 3''.9$. The double altitude observed was, therefore, $105° \ 16' \ 7''.8$. The index correction of the sextant was $- 3' \ 30''$, and hence the double altitude given by the instrument was $105° \ 15' \ 25''$, which was, consequently, too small by $43''$.

To compute the differential equations, we find

$$A = 293° \ 45'.2 \quad A' = 182° \ 9'.1 \quad A'' = 90° \ 17'.9$$

and hence

$$d\varphi = + \ 3.808 \ dT - 0.288 \ dT' - 3.519 \ dT'''$$
$$d\Delta T = - \ 0.391 \ dT - 0.007 \ dT' - 0.602 \ dT'''$$

by which we see that an error of one second in each of the times would produce at the most but $7''.6$ error in the latitude, and one second in the clock correction.

188. *Solution of the preceding problem by* CAGNOLI's *formulæ.*— After GAUSS had published the solution above given, he was himself the first to observe[*] that CAGNOLI's formulæ for the solution of a very different problem[†] might be applied directly to this.

When the altitude is also computed, CAGNOLI's formulæ have slightly the advantage over those of GAUSS. To deduce them, let q, q', q'' be the parallactic angles at the three stars, or (Fig. 26) let

Fig. 26. (*bis*).
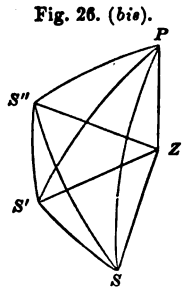
$$q = PSZ, \qquad q' = PS'Z, \qquad q'' = PS''Z,$$

and also put

$$Q \ = \tfrac{1}{2}(PS''S' - PS'S'')$$
$$Q' \ = \tfrac{1}{2}(PS''S \ - PSS'')$$
$$Q'' = \tfrac{1}{2}(PS'S \ - PSS')$$

---

[*] *Monatliche Correspondenz*, Vol. XIX. p. 87.

[†] Namely, that of determining, from three heliocentric places of a solar spot, the position of the sun's equator, and the declination of the spot.—See CAGNOLI's *Trigonométrie*, p. 488.

Substituting the values of q and q' in terms of Q, this gives

$$\tan (t + \tfrac{1}{2}\lambda) = \tan \tfrac{1}{2}\lambda \tan Q'' \cot (Q - Q')$$

or, substituting the value of $\tan Q''$,

$$\tan (t + \tfrac{1}{2}\lambda) = \frac{\sin \tfrac{1}{2}(\delta' - \delta)}{\cos \tfrac{1}{2}(\delta' + \delta)} \cot (Q - Q') \qquad (347)$$

which determines $t + \tfrac{1}{2}\lambda$, whence t and the clock correction. We can now find the latitude and altitude from any one of the triangles PSZ, $PS'Z$, $PS''Z$, by NAPIER's Analogies (Sph. Trig. Art. 80): thus, from PSZ we have

$$\left. \begin{aligned} \tan \tfrac{1}{2}(\varphi + h) &= \frac{\cos \tfrac{1}{2}(t + q)}{\cos \tfrac{1}{2}(t - q)} \tan (45° + \tfrac{1}{2}\delta) \\[2mm] \tan \tfrac{1}{2}(\varphi - h) &= \frac{\sin \tfrac{1}{2}(t - q)}{\sin \tfrac{1}{2}(t + q)} \cot (45° + \tfrac{1}{2}\delta) \end{aligned} \right\} \qquad (348)$$

and then $\varphi = \tfrac{1}{2}(\varphi + h) + \tfrac{1}{2}(\varphi - h)$, $h = \tfrac{1}{2}(\varphi + h) - \tfrac{1}{2}(\varphi - h)$.

As all the angles are determined by their tangents, an ambiguity exists as to the semicircle in which they are to be taken; but, as GAUSS remarks, we may choose arbitrarily (taking, for example, Q, Q', Q'' always less than 90°, positive or negative according to the signs of their tangents), and then, according to the results, will have in some cases to make the following changes:

    1. If the values of φ and h found by (348) are such that $\cos \varphi$ and $\sin h$ have opposite signs, we must substitute $180° + q$ for q and repeat the computation of these two equations. In this repetition the same logarithms will occur as before, but differently placed.

    2. If the values of φ and h exceed 90°, we must take their supplements to the next multiple of 180°.

    3. The latitude is to be taken as north or south according as $\sin \varphi$ and $\sin h$ have the same or different signs.

No ambiguity, however, exists in practice as to $t + \tfrac{1}{2}\lambda$, found by (347), since $Q - Q'$ can differ from its true value only by 180°, and this difference does not change the sign of $\cot (Q - Q')$: hence $\tan (t + \tfrac{1}{2}\lambda)$ will come out with its true sign; and between

these equations, dh and $d\varphi$ are the same, but dt is different for each. If 'we put

$$\left.\begin{array}{l} f \;= T \; + \vartriangle T + \delta T \, (T \; - T_0) - (\alpha \; + t \;) \\ f' = T' \; + \vartriangle T + \delta T \, (T' \; - T_0) - (\alpha' + t') \\ f'' = T'' + \vartriangle T + \delta T \, (T'' - T_0) - (\alpha'' + t'') \\ \qquad \&\text{c.} \end{array}\right\} \quad (349)$$

which are all known quantities, we have

$$dt = f + d \vartriangle T, \qquad dt' = f' + d \vartriangle T, \&\text{c.}$$

and the equations of condition become

$$\left.\begin{array}{l} dh + \cos A \;.\, d\varphi + 15 \cos \varphi \sin A \;.\, d \vartriangle T + 15 \cos \varphi \sin A \;.\, f \;= 0 \\ dh + \cos A'.\, d\varphi + 15 \cos \varphi \sin A'.\, d \vartriangle T + 15 \cos \varphi \sin A'.\, f' = 0 \\ dh + \cos A''.\, d\varphi + 15 \cos \varphi \sin A''.\, d \vartriangle T + 15 \cos \varphi \sin A''.\, f'' = 0 \\ \qquad \&\text{c.} \end{array}\right\} \quad (350)$$

from which, by the method of least squares, the most probable values of dh, $d\varphi$, and $d \vartriangle T$ are determined. The true values of the altitude, latitude, and clock correction will then be $h + dh$, $\varphi + d\varphi$, $\vartriangle T + d \vartriangle T$.

The hour angles will be computed most accurately by (269), which is the same as the following:

$$\tan^2 \tfrac{1}{2} t = \frac{\sin \tfrac{1}{2} \, (\zeta - \varphi + \delta) \, \sin \tfrac{1}{2} \, (\zeta + \varphi - \delta)}{\cos \tfrac{1}{2} \, (\zeta + \varphi + \delta) \, \cos \tfrac{1}{2} \, (\zeta - \varphi - \delta)}$$

in which $\zeta = 90° - h$; and the azimuths by

$$\tan^2 \tfrac{1}{2} A = \frac{\sin \tfrac{1}{2} \, (\zeta - \varphi + \delta) \, \cos \tfrac{1}{2} \, (\zeta - \varphi - \delta)}{\cos \tfrac{1}{2} \, (\zeta + \varphi + \delta) \, \sin \tfrac{1}{2} \, (\zeta + \varphi - \delta)}$$

Since φ and ζ are constant, it will be convenient to put

$$b = \tfrac{1}{2} \, (\zeta + \varphi) \qquad\qquad c = \tfrac{1}{2} \, (\zeta - \varphi)$$

$$m = \frac{\sin \, (c + \tfrac{1}{2} \, \delta)}{\cos \, (b + \tfrac{1}{2} \, \delta)} \qquad\qquad n = \frac{\sin \, (b - \tfrac{1}{2} \, \delta)}{\cos \, (c - \tfrac{1}{2} \, \delta)}$$

then

$$\tan^2 \tfrac{1}{2} t = mn \qquad \tan^2 \tfrac{1}{2} A = \frac{m}{n} \qquad\qquad (351)$$

The barometer and thermometer should be observed with each

altitude, and if they indicate a sensible change in the refraction a correction for this change must be introduced into the equations of condition. Thus, if r_0 is the refraction for the altitude h for the mean height of the barometer and thermometer during the whole series, while for one of the stars it is r, then the assumed altitude requires for that star not only the correction dh, but also the correction $r - r_0$. Hence, if we find the refractions r, r', r'', &c. for all the observations, and take their mean r_0, we have only to add to the equations of condition respectively the quantities $r - r_0$, $r' - r_0$, $r'' - r_0$, &c.

If any one of the stars is observed at an altitude h_1 slightly different from the common altitude h, we correct the corresponding equation of condition by adding the quantity $h - h_1$.

190. We may also apply the preceding method to the case where there are but three observations. The final equations are then nothing more than the three equations of condition themselves, from which the unknown quantities will be found by simple elimination. It will easily be seen that this elimination leads to the expressions for $d\varphi$ and $d\Delta T$ already given on p. 284, if we there exchange dT, dT', and dT'' for f, f', and f'' respectively. We can simplify the computation by assuming ΔT so as to make one of the quantities f, f', f'' zero. Thus, we shall have $f = 0$ if we determine ΔT by the formula

$$\Delta T = a + t - [T + \delta T (T - T_0)] \qquad (352)$$

then, finding f' and f'' with this value, and putting

$$k' = \frac{\sin \tfrac{1}{2} A' \cos \tfrac{1}{2} A'}{\sin \tfrac{1}{2} (A' - A) \sin \tfrac{1}{2} (A'' - A')} \cdot f'$$

$$k'' = \frac{\sin \tfrac{1}{2} A'' \cos \tfrac{1}{2} A''}{\sin \tfrac{1}{2} (A'' - A) \sin \tfrac{1}{2} (A'' - A')} \cdot f''$$

we shall have the following formulæ:

$$\left. \begin{aligned} d\Delta T &= - k' \sin \tfrac{1}{2} (A + A'') + k'' \sin \tfrac{1}{2} (A' + A) \\ \frac{d\varphi}{15 \cos \varphi} &= - k' \cos \tfrac{1}{2} (A + A'') + k'' \cos \tfrac{1}{2} (A' + A) \\ \frac{dh}{15 \cos \varphi} &= + k' \cos \tfrac{1}{2} (A'' - A) - k'' \cos \tfrac{1}{2} (A' - A) \end{aligned} \right\} \qquad (353)$$

EXAMPLE.—Taking the three observations above employed, and assuming the approximate values

$$\Delta T = - 11^m\ 0^s, \quad \varphi = 51°\ 32'\ 0'', \quad h = 52°\ 37'\ 0'',$$

we shall find, by (351),

$$t = -\ 2^h\ 36^m\ 5^s.50 \quad t' = -\ 3^h\ 19^m\ 55^s.65 \quad t'' = 3^h\ 23^m\ 58^s.25$$
$$A = -\ 66°\ 15'.2 \quad\quad A' = -\ 177°\ 50'.2 \quad\quad A'' = 90°\ 18'.1$$

By (349), putting in this case $\delta T = 0$, we then have

$$f = -\ 1^s.83 \quad\quad f' = +\ 80^s.95 \quad\quad f'' = -\ 6^s.21$$

and the equations of condition (350) become

$$dh + 0.4027\ d\varphi - 8.5410\ d\Delta T + 15.63 = 0$$
$$dh - 0.9993\ d\varphi - 0.3522\ d\Delta T - 28.51 = 0$$
$$dh - 0.0053\ d\varphi + 9.3308\ d\Delta T - 57.94 = 0$$

whence

$$d\Delta T = +\ 3^s.92 \quad\quad d\varphi = -\ 8''.58 \quad\quad dh = +\ 21''.31$$

and the true values of the required quantities are, therefore,

$$\Delta T = -\ 10^m\ 56^s.08 \quad\quad \varphi = 51°\ 31'\ 51''.42 \quad\quad h = 52°\ 37'\ 21''.31$$

agreeing almost perfectly with the values before found.

Since in this example there are but three observations, we may also employ the formulæ (353), first assuming

$$\Delta T = -\ 10^m\ 58^s.17$$

which is the value given by (352). With this we find

$$f' = +\ 82^s.78 \quad\quad f'' = -\ 4^s.38$$
$$\log k' = 0.4199 \quad\quad \log k'' = n0.4932$$

and by (353) we shall find

$$d\Delta T = +\ 2^s.09 \quad\quad d\varphi = -\ 8''.58 \cdot \quad dh = +\ 21''.31$$

Hence the true clock correction is $-\ 10^m\ 58^s.17 + 2^s.09 = -\ 10^m\ 56^s.08$; and the values of the latitude and altitude also agree with the former values.

and then, the declination of the star being given, we have the equation [from (14)]

$$\cos t \sin \varphi - \tan \delta \cos \varphi = \sin t \cot A \qquad (354)$$

If, then, A is also known, the latitude φ can be found by this equation. Let us inquire under what conditions an accurate result is to be expected by this method. By differentiating the equation, we find [see (51)]

$$d\varphi = \frac{\cos q \cos \delta}{\cos \zeta \sin A} dt - \frac{\tan \zeta}{\sin A} dA + \frac{\sin q}{\cos \zeta \sin A} d\delta$$

from which it appears that sin A and cos ζ must be as great as possible. The most favorable case is, therefore, that in which the assumed vertical circle is the *prime vertical*, and the star's declination differs but little from the latitude; for we then have $A = 90°$ and ζ small. Indeed, these conditions not only increase the denominator of the coefficient of dt, but also diminish its numerator, since, by (10), we have

$$\cos q \cos \delta = \sin \zeta \sin \varphi + \cos \zeta \cos \varphi \cos A$$

which vanishes wholly when the star passes through the zenith. Moreover, if the same star is observed at both its east and west transits over the prime vertical, we shall have at one transit sin $A = - 1$, at the other sin $A = + 1$, and the mean of the two resulting values of the latitude will, therefore, be wholly free from the effect of a constant error in the clock times, that is, of an error in the clock correction. It is then necessary only that the *rate* should be known. This method, therefore, admits of a high degree of precision, and requires for its successful application only a transit instrument, of moderate dimensions, and a time-piece. Its advantages were first clearly demonstrated by BESSEL* in the year 1824; but it appears that very early in the last century RÖMER had mounted a transit instrument in the prime vertical for the purpose of determining the declinations of stars from their transits, the latitude being given. The details of this important method will be given in Vol. II., under " Transit Instrument."

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* *Astronom. Nach.*, Vol. III. p. 9.

193. It may sometimes be possible to observe transits only over some vertical circle the azimuth of which is undetermined. We must then observe either two stars, or the same star on opposite sides of the meridian. We shall then have the two equations

$$\cos t \,.\, \tan A \sin \varphi - \tan \delta \,.\, \tan A \cos \varphi = \sin t$$
$$\cos t' \,.\, \tan A \sin \varphi - \tan \delta' \,.\, \tan A \cos \varphi = \sin t'$$

from which the two unknown quantities A and φ can be determined. If the same star is observed, we shall only have to put $\delta' = \delta$. Regarding $\tan A \sin \varphi$ and $\tan A \cos \varphi$ as the unknown quantities, we have, by eliminating them in succession,

$$\tan A \sin \varphi = \frac{\sin \quad \sin \delta' \cos \delta - \sin t' \cos \delta' \sin \delta}{\cos t \sin \delta' \cos \delta - \cos t' \cos \delta' \sin \delta}$$

$$\tan A \cos \varphi = \frac{- \sin (t' - t) \cos \delta' \cos \delta}{\cos t \sin \delta' \cos \delta - \cos t' \cos \delta' \sin \delta}$$

If we introduce the auxiliaries m and M, such that

$$\left. \begin{array}{l} m \sin M = \sin (\delta' + \delta) \sin \tfrac{1}{2} (t' - t) \\ m \cos M = \sin (\delta' - \delta) \cos \tfrac{1}{2} (t' - t) \end{array} \right\} (355)$$

we shall easily find

$$m \sin [\tfrac{1}{2} (t' + t) - M] = \sin t \sin \delta' \cos \delta - \sin t' \cos \delta' \sin \delta$$
$$m \cos [\tfrac{1}{2} (t' + t) - M] = \cos t \sin \delta' \cos \delta - \cos t' \cos \delta' \sin \delta$$
$$m \sin [\tfrac{1}{2} (t' - t) - M] = - \sin (t' - t) \cos \delta' \sin \delta$$

and hence

$$\left. \begin{array}{l} \tan A \sin \varphi = \tan [\tfrac{1}{2} (t' + t) - M] \\[2mm] \tan A \cos \varphi = \dfrac{\sin [\tfrac{1}{2} (t' - t) - M] \cot \delta}{\cos [\tfrac{1}{2} (t' + t) - M]} \end{array} \right\} (356)$$

which determine A and φ by a simple logarithmic computation. The solution will be still more convenient in the following form:

$$\left. \begin{array}{l} \tan M = \tan \tfrac{1}{2} (t' - t) \dfrac{\sin (\delta' + \delta)}{\sin (\delta' - \delta)} \\[4mm] \tan \varphi = \tan \delta \dfrac{\sin [\tfrac{1}{2} (t' + t) - M]}{\sin [\tfrac{1}{2} (t' - t) - M]} \\[4mm] \tan A = \dfrac{\tan [\tfrac{1}{2} (t' + t) - M]}{\sin \varphi} \end{array} \right\} (357)$$

If the same star is observed at each of its transits over the same vertical circle, we have $\delta' = \delta$, and hence $\tan M = \infty$, $M = 90°$, which gives

$$\tan \varphi = \tan \delta \frac{\cos \frac{1}{2}(t' + t)}{\cos \frac{1}{2}(t' - t)} \qquad \tan A = -\frac{\cot \frac{1}{2}(t' + t)}{\sin \varphi} \quad (358)$$

If the same star is observed twice on the prime vertical, we must have $t' + t = 0$, since $\tan A = \infty$; and then,

$$\tan \varphi = \frac{\tan \delta}{\cos \frac{1}{2}(t' - t)} = \frac{\tan \delta}{\cos t} \qquad (359)$$

which follows also from (354) when $\cot A = 0$; or, geometrically, from the right triangle formed by the zenith, the pole, and the star, as in Art. 19.

If the latitude is given, we can find the time from the transits of two stars over any (undetermined) vertical circle by the second equation of (357), which gives

$$\sin [\tfrac{1}{2}(t' + t) - M] = \frac{\tan \varphi}{\tan \delta} \sin [\tfrac{1}{2}(t' - t) - M]$$

for the observation furnishes the elapsed time, and hence $t' - t$; and this equation determines $\frac{1}{2}(t' + t)$, and hence both t and t'.

If the latitude and time are given, we can find the declination of a star observed twice on the same vertical circle, by (358). When the observation is made in the prime vertical, this becomes one of the most perfect methods of determining declinations. See Vol. II., *Transit Instrument in the Prime Vertical.*

194. The following brief approximative methods of determining the latitude may be found useful in certain cases.

TENTH METHOD.—BY ALTITUDES NEAR THE MERIDIAN WHEN THE TIME IS NOT KNOWN.

195. (A.) *By two altitudes near the meridian and the chronometer times of the observations, when the rate of the chronometer is known, but not its correction.*
   Let

$$h, \ h' = \text{the true altitudes,}$$
$$T, \ T' = \text{the chronometer times,}$$
$$\tau = \tfrac{1}{2}(T' - T)$$

then, t and t' being the (unknown) hour angles of the observations, we have, by (287), approximately,

$$h_1 = h + at^2$$
$$h_1 = h' + at'^2$$

in which h_1 is the meridian altitude, and

$$a = \frac{225 \sin 1''}{2} \cdot \frac{\cos \varphi \cos \delta}{\cos h_1}$$

The mean of these equations is

$$h_1 = \tfrac{1}{2}(h + h') + a\left[\left(\frac{t'-t}{2}\right)^2 + \left(\frac{t'+t}{2}\right)^2\right]$$

and their difference gives

$$h - h' = a(t' - t)(t' + t)$$

But we have

$$\tau = \tfrac{1}{2}(T' - T) = \tfrac{1}{2}(t' - t)$$

in which we suppose the interval $T' - T$ to be corrected for the rate of the chronometer.   Hence

$$\frac{t' + t}{2} = \frac{\tfrac{1}{2}(h - h')}{a\tau}$$

which, substituted in the above expression for h_1, gives

$$h_1 = \tfrac{1}{2}(h + h') + a\tau^2 + \frac{[\tfrac{1}{2}(h - h')]^2}{a\tau^2} \qquad (360)$$

According to this formula, the mean of the two altitudes is reduced to the meridian by adding two corrections: 1st, the quantity $a\tau^2$, which is nothing more than the common "reduction to the meridian" computed with the half elapsed time as the hour angle; 2d, the square of one-fourth the difference of the altitudes divided by the first correction.

If we employ the form (285) for the reduction, we have

$$h_1 = \tfrac{1}{2}(h + h') + Am + \frac{[\tfrac{1}{2}(h - h')]^2}{Am} \qquad . \qquad (361)$$

in which

$$A = \frac{\cos \varphi \cos \delta}{\cos h_1} \qquad\qquad m = \frac{2 \sin^2 \tfrac{1}{2}\tau}{\sin 1''}$$

and m is taken from Table V. or log m from Table VI.

EXAMPLE 1.—From the observations in the example of Art. 171, I select the following, which are very near the meridian.

Obsd. alts. ☉		True alts. ☉	Chronometer.
50° 5′ 42″.8	$h' =$	50° 21′ 7″.6	23ʰ 50ᵐ 46ˢ.5
50 7 25 .5	$h =$	50 22 50 .4	0 0 37 .5
$\frac{1}{4}(h - h') =$		26 .1	$\tau =$ 4 55 .5
$\frac{1}{2}(h + h') =$		50 21 59 .0	
$Am =$		+ 59 .2	log m 1.6790
2d corr. $=$		+ 10 .6	log A 0.0930
$h_1 =$		50 23 8 .8	log Am 1.7720
$\zeta_1 =$		39 36 51 .2	log $[\frac{1}{4}(h - h')]^2$ 2.7958
$\delta_1 =$		$-$ 1 48 9 .2	log 2d corr. 1.0238
$\varphi =$		37 48 42 .0	

EXAMPLE 2.—In the same example, the first and last observations, which are quite remote from the meridian, are as follows:

Obsd. alts. ☉		True alts. ☉	Chronometer.
49° 51′ 19″.3	$h =$	50° 6′ 43″.7	23ʰ 37ᵐ 35ˢ
49 50 24	$h' =$	50 5 48 .4	0 18 31
$\frac{1}{4}(h - h') =$		13 .8	$\tau =$ 20 28

which give $Am = 16′ 59″$, and the 2d corr. $= 0″.2$, whence $\varphi = 37° 48′ 37″$.

This simple approximative method may frequently be useful to the traveller, and especially at sea, where the meridian observation has been lost in consequence of flying clouds. At sea, however, the computation need not be carried out so minutely as the above, and the method becomes even more simple. See Art. 204.

M. V. CAILLET[*] gives a method for the same purpose, which is readily deduced from the above. Put

$$k = h' - h \qquad \tau' = T' - T = 2\tau$$

then (360) becomes

$$h_1 = h + \frac{k}{2} + \frac{a\tau'^2}{4} + \frac{k^2}{4a\tau'^2}$$
$$= h + \frac{(k + a\tau'^2)^2}{4a\tau'^2}$$

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[*] *Traité de Navigation* (2d edition, Paris, 1857), p. 819.

or, putting

$$m = \frac{2 \sin^2 \tfrac{1}{2} \tau'}{\sin 1''} \qquad Am = a \tau'^2$$

$$h_1 = h + \frac{(k + Am)^2}{4\,Am} \tag{362}$$

in which h is the altitude farthest from the meridian. Although this reduces the two corrections of (361) to a single one, the computation is not quite so simple.

196. (B.) *By three altitudes near the meridian and the chronometer times of the observations, when neither the correction nor the rate of the chronometer is known.*—In this case we assume only that the chronometer goes *uniformly* during the time occupied by the observations.   Let

$$
\begin{aligned}
h, \ h', \ h'' \ &= \text{the true altitudes,}\\
T, \ T', \ T'' \ &= \text{the chronometer times,}\\
T_1 \ &= \text{the chronometer time of the greatest altitude.}
\end{aligned}
$$

If we introduce the factor for rate $= k$, according to Art. 171, the formula for the reduction to the meridian by GAUSS's method is, approximately,

$$h_1 = h + ak t^2$$

in which t is the time reckoned from the greatest altitude.   Denoting ak by a, we have then, from the three observations,

$$
\left.
\begin{aligned}
h_1 &= h + a\,(T \ - T_1)^2\\
h_1 &= h' + a\,(T' - T_1)^2\\
h_1 &= h'' + a\,(T'' - T_1)^2
\end{aligned}
\right\} \tag{363}
$$

which three equations suffice to determine the three unknown quantities a, T_1, and h_1. By subtracting the second from the first, and the third from the second, we obtain

$$\frac{h \ - h'}{T' - T} = a\,(T' + T) - 2a T_1$$

$$\frac{h' \ - h''}{T'' - T'} = a\,(T'' + T') - 2a T_1$$

and the difference of these is

$$\frac{h' - h''}{T'' - T'} - \frac{h - h'}{T' - T} = a\,(T'' - T)$$

197. (C.) *By two altitudes or zenith distances near the meridian and the difference of the azimuths.*—If the observer has no chronometer, he may still obtain his latitude by circummeridian altitudes, if he observes the altitudes with a universal instrument, and reads the horizontal circle at each observation, taking care, of course, that the star is always observed at the middle vertical thread. As this instrument generally gives directly the zenith distances, we shall substitute ζ for $90° - h$. We have the equation

$$\sin \delta = \sin \varphi \cos \zeta - \cos \varphi \sin \zeta \cos A$$
$$= \sin (\varphi - \zeta) + 2 \cos \varphi \sin \zeta \sin^2 \tfrac{1}{2} A$$

whence

$$\cos \tfrac{1}{2} (\varphi + \delta - \zeta) \sin \tfrac{1}{2} [\zeta - (\varphi - \delta)] = \cos \varphi \sin \zeta \sin^2 \tfrac{1}{2} A$$

But

$$\varphi - \delta = \zeta_1 = \text{the meridian zenith distance;}$$

and hence

$$\sin \tfrac{1}{2} (\zeta - \zeta_1) = \frac{\cos \varphi \sin \zeta \sin^2 \tfrac{1}{2} A}{\cos [\delta - \tfrac{1}{2}(\zeta - \zeta_1)]} \qquad (365)$$

which expresses the reduction to the meridian $= \zeta - \zeta_1$ when the absolute azimuth A is given. If the observation is very near the meridian, we may neglect $\tfrac{1}{2} (\zeta - \zeta_1)$ in the denominator of the second member, and take

$$\zeta - \zeta_1 = \frac{\cos \varphi \sin \zeta_1}{\cos \delta} \cdot \frac{2 \sin^2 \tfrac{1}{2} A}{\sin 1''}$$

or, putting

$$a = \frac{\cos \varphi \sin \zeta_1}{\cos \delta} \cdot \frac{\sin 1''}{2} \qquad (366)$$

$$\zeta - \zeta_1 = aA^2 \qquad (367)$$

from which it follows that near the meridian the zenith distance varies as the square of the azimuth.

Now, when we have taken two observations, we have

$$\zeta_1 = \zeta - aA^2$$
$$\zeta_1 = \zeta' - aA'^2$$

whence, putting

$$\tau = \tfrac{1}{2} (A' - A)$$

its image, the sextant reading being the same at both observations, namely, 30° 15′ 0″. He found

<div align="center">
Chronometer.
</div>

Contact of lower limb,	4^h 43^m 34^s.	P.M.
" upper "	4 47	5 .5
	3 31 .5	

The sun's diameter was 31′ 32″. Hence we have

$d\zeta = 31′\ 32″\ = 1892″$	log	3.2769
$dt = 3^m\ 31^s.5 = 211^s.5$	ar. co. log	7.6747
	$\log \frac{1}{15}$	8.8239
$\varphi = 53°\ 23′.5$	log cos φ	9.7755

The azimuth, however, was not exactly 90°, but about 88° 20′. Hence we shall have, more exactly,

		9.7755
$A = 88°\ 20′$	log cosec A	0.0002
$\varphi = 53\ \ 22.3$	log cos φ	9.7757

It is evident that the method will be more precise in high latitudes than in low ones.

<div align="center">FINDING THE LATITUDE AT SEA.</div>

First Method.—By Meridian Altitudes.

200. This is the most common, as well as the simplest and most reliable, of the methods used by the navigator. The altitude is observed with the sextant (or quadrant) from the sea horizon, and, in addition to the corrections used on shore, the dip of the horizon is to be applied. The true altitude being deduced, the latitude is found by (277) or (278), Art. 161.

At sea the time is seldom so well known as to enable the navigator to take the star at the precise instant of its meridian passage. But the meridian altitude of a star is distinguished as the *greatest*, to secure which the observer commences to measure the star's altitude some minutes before the approximately computed time of passage, and continues to observe it until he perceives it to be falling. The greatest of all his measures is then assumed as the meridian altitude.

The most common practice in the case of the sun is to bring the lower limb, reflected in the mirrors of the instrument, to touch the sea horizon seen directly (a few minutes before noon), and then by the tangent screw to follow the sun as long as it rises, never reversing the motion of the screw; as soon as the sun begins to fall, the limb will appear "to dip" in the sea by lapping over the line marking the horizon. Hence, when the sun "dips," the observation is complete, and the instrument is read off. But, as the waves of the sea cause the ship to rise and fall, the depression of the sea horizon is constantly fluctuating by the small amount due to the change in the height of the observer's eye: it is, consequently, impossible to keep the sun's reflected image in *constant* contact with the horizon. Experienced observers advise, therefore, to observe and read off separate altitudes in rapid succession, continuing until the numbers read off decidedly decrease; the greatest is then taken as the meridian altitude,* or, still more accurately, the mean of the greatest and the two immediately adjacent may be taken as the meridian altitude, free from the inequalities produced by the motion of the eye.

201. The greatest altitude, however, is not the meridian altitude, except in the case of a fixed star. To find the correction for a change of declination, we have, for the time (ϑ) from noon when the sun is at the greatest altitude, the formula (294), or

$$\vartheta = \frac{\Delta \delta}{810000 \sin 1''} \cdot \frac{\sin (\varphi - \delta)}{\cos \varphi \cos \delta}$$

in which $\Delta\delta$ is the hourly change of declination expressed in seconds. The reduction of the maximum altitude to the meridian altitude is the quantity y, Art. 172, or

$$y = \frac{(15 \vartheta)^2 \sin 1''}{2} \cdot \frac{\cos \varphi \cos \delta}{\sin (\varphi - \delta)}$$

These formulæ give ϑ in seconds of time and y in seconds of arc. For nautical use, let

$a =$ the change of altitude (expressed in seconds of arc) in one minute of time from the meridian;

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* Raper, *Practice of Navigation* (4th edition, 1852), p. 226.

or due south at the rate of 18 miles per hour, the maximum altitude will exceed the meridian altitude by 2′ 15″.

Second Method.—By Reduction to the Meridian when the Time is given.

203. When the meridian observation is lost in consequence of clouds, circummeridian altitudes may sometimes be obtained. The most convenient method of reducing them at sea is that of BOWDITCH. In his Table XXXII. he gives the value of a computed by (373); and in Table XXXIII. the value of t^2, t being reduced to minutes. Each observed altitude h is then reduced to the meridian altitude h_1 by the formula (287), or

$$h_1 = h + at^2 \tag{375}$$

and a number of altitudes are reduced at once by the same formula, by taking for h the mean of all the altitudes, and for t^2 the mean of all the values of t^2. If the observer has no tables, he can readily compute a by the formula

$$a = 1''.9635 \, \frac{\cos \varphi \cos \delta}{\sin (\varphi - \delta)} = [0.2930] \, \frac{\cos \varphi \cos \delta}{\sin (\varphi - \delta)} \tag{376}$$

BOWDITCH's table for t^2 extends, however, only to $t = 13^m$. When the observations are more than 13^m from the meridian, he reduces the observation to the meridian by the formula (282),

$$\cos \zeta_1 = \sin h + \cos \varphi \cos \delta \, (2 \sin^2 \tfrac{1}{2} t)$$

employing a table of log. versed sines for the value of $2 \sin^2 \tfrac{1}{2} t$; a table of natural sines for $\sin h$ and $\cos \zeta_1$; and the table of logarithms of numbers for the value of the last term. I prefer the formula (283),

$$\sin \tfrac{1}{2} (h_1 - h) = \frac{\cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t}{\cos \tfrac{1}{2} (h_1 + h)}$$

which effects the reduction by a single table.

Third Method.—By Two Altitudes near the Meridian when the Time is not known.

204. As it frequently happens at sea that the local time is uncertain, the method I have proposed in Art. 195 will be found

of great use to the navigator.   Any two altitudes h and h' being observed near the meridian, τ being *one-half* the chronometer interval between them, corrected for rate, expressed in minutes, and a being found by (376), or from BOWDITCH's Table XXXII, we have the meridian altitude by the formula

$$h_1 = \tfrac{1}{2}(h + h') + a\tau^2 + \frac{[\tfrac{1}{2}(h - h')]^2}{a\tau^2}$$

which may be computed without the use of logarithms.

EXAMPLE.—The approximate latitude being 38° N., the declination at noon 1° 48′ 9″ S., the height of the eye above the sea 19 feet, suppose the following observations taken:

	Chronometer.			☉		
$T' =$	8ʰ 0ᵐ 22ˢ.5		$h' =$	50°	10′	0″
$T =$	8 10 13.5		$h =$	50	11	40
	2) 9 51		$h - h' =$		1	40
$\tau =$	4 55.5	$\tfrac{1}{4}(h - h') =$				25
$\tau^2 =$	24.2	$\tfrac{1}{2}(h + h') =$	50	10	50	
$a =$	2″.4	$a\tau^2 =$ 1st corr. $=$			58	
$[\tfrac{1}{4}(h - h')]^2 =$	625	$\frac{625}{58} =$ 2d " $=$			11	
		Merid. alt. ☉ $=$	50	11	59	
		Dip $= -$		4	16	
		Semidiameter $= +$	16	6		
		Refr. and par. $= -$			42	
		$h_1 =$	50	23	7	
		$\zeta_1 =$	39	36	53	N.
		$\delta_1 =$	1	48	9	S.
		$\varphi =$	37	48	44	N.

The accuracy of the result depends in a great degree upon the accuracy with which the *difference* of altitude is obtained. If in the above example this difference had been 2′ 40″, or 1′ too great, we should have found $\tfrac{1}{4}(h - h') = 40$″, and the 2d correction $= \frac{1600}{58} = 28$″: consequently the resulting latitude would have been only 17″ too small.   Since the same causes of error, such as displacement of the sea horizon by extraordinary refraction, unknown instrumental errors, &c., affect both altitudes alike, the difference will usually be obtained, even at sea, within a quantity much less than 1′.   The most favorable case is that

in which the altitudes are equal and the 2d correction, consequently, zero. It will be well, therefore, always to endeavor to obtain altitudes on opposite sides of the meridian.

We may also obtain an approximate value of the time from the same observation; for we have for the hour angle of the least altitude h', Art. 195,

$$t' = \frac{'t + t}{2} + \tau$$

or

$$t' = \frac{\frac{1}{4}(h - h')}{a\tau} + \frac{1}{2}(T' - T)$$

Thus, in the above example we have

$$\frac{\frac{1}{4}(h - h')}{a\tau} = \frac{25}{2.4 \times 4.9} = \quad 2^m.1$$

$$\frac{1}{2}(T' - T) = + 4 .9$$

$$t' = + 7 .0$$

The apparent time of the observation of the least altitude was, therefore, $0^h 7^m$.

Fourth Method.—By Three Altitudes near the Meridian when the Time is not known.

205. The method of Art. 196 does not require even the rate of the chronometer to be known; but it is hardly simple enough for a common nautical method. But a very simple method will be obtained if we take *three altitudes at equal intervals of time.* Suppose the second altitude is observed at the (unknown) time T from the meridian passage, the first at the time $T - x$, the third at the time $T + x$; then we have, by (363),

$$h_1 = h \;\mp\; a (T - x)^2$$
$$h_1 = h' + a T^2$$
$$h_1 = h'' + a (T + x)^2$$

Subtracting the half sum of the first and third equations from the second, we deduce

$$ax^2 = h' - \frac{1}{2}(h + h'')$$

the error in the latitude produced by an error in the time increases very rapidly as the star leaves the meridian and approaches the prime vertical (Art. 166), and the method fails altogether when the star is in the prime vertical. It may, however, sometimes be very important to determine the latitude, at least approximately, when the sun is nearly east or west; and then the following method may be used.

Sixth Method.—By the change of Altitude near the Prime Vertical.

207. This is the method of Art. 199. In the morning, when the sun has arrived within 1° of the prime vertical as observed with the ship's compass, bring the image of the sun's upper limb, reflected by the sextant mirrors, into contact with the sea horizon, and note the time; let the sextant reading remain unchanged, and note the time when the contact of the lower limb occurs. In the afternoon, begin with the lower limb. Then, taking the sun's semidiameter $= S$ from the almanac, and putting the difference of the chronometer times $= \tau$, we have

$$\cos \varphi = \frac{2S}{15\tau} = [9.1249]\frac{S}{\tau} \tag{378}$$

This is evidently but a rough method, only to be resorted to in cases of emergency. With the greatest care in observing the contacts, and in latitudes not less than 45°, the result cannot be depended upon within from five to ten minutes; but even this degree of accuracy may, in many cases at sea, be quite satisfactory.

Seventh Method.—By the Pole Star.

208. This method, though confined in its application to north latitudes, is very useful at sea, as it is available at all times when the star is visible and the horizon sufficiently distinct, and does not require a more accurate knowledge of the time than is usually possessed on shipboard. The complete discussion of it has been given in Art. 176; but for those who wish only the nautical method, and have passed over that article, I add the following simple investigation, which is sufficiently precise for the purpose.

Let ZN, Fig. 27, be the meridian; Z the zenith of the observer; P the pole; AN the horizon; S the star, which describes

Fig. 27.
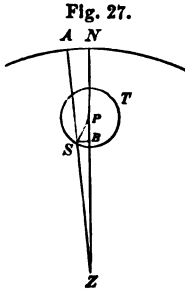
a small circle ST about the pole at the distance $PS = p$; ZSA the vertical circle of the star at the time of the observation; SA the true altitude $= h$, deduced from the observed; SPZ the star's hour angle $= t$; PN the latitude $= \varphi$.

Draw SB perpendicular to the meridian: then, since SP is small in the case of the pole star (about 1° 30′), we may regard PSB as a plane triangle, and hence we have

$$PB = PS \cdot \cos SPB = p \cos t$$

and, since BN differs very little from SA,

$$PN = BN - PB = SA - PB$$

that is,*

$$\varphi = h - p \cos t$$

If we put

$\Theta =$ the sidereal time,
$a =$ the star's right ascension,

we have

$$t = \Theta - a$$

and hence

$$\varphi = h - p \cos (\Theta - a) \qquad (379)$$

If then p and α be regarded as constant, the term $p \cos (\Theta - \alpha)$ may be given in a table with the argument Θ, as in BOWDITCH'S Navigator, p. 206. But the polar distance and right ascension of the pole star vary so rapidly that in a few years such a table affords but a rude approximation. The direct computation of the formula with the values of p and α obtained from the Ephemeris for the day of the observation is preferable.

EXAMPLE.—1856 March 10, from an altitude of *Polaris* observed from the sea horizon, the true altitude h was deduced as below. The time was noted by a Greenwich chronometer which was fast 5ᵐ 30ˢ. The longitude was 150° 0′ W.

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* If we compare this with the more exact formula (300), we see that the error of the nautical method is $\frac{1}{2} p^2 \sin 1'' \sin^2 t \tan h$, which is a maximum for $t = 90°$. Taking $p = 1° 30′$, this maximum is $70''.7 \tan \phi$, which amounts to 8′ when $\phi = 68° 30′$.

Chronometer	19ʰ 12ᵐ 42ˢ					$h = 31°\ 10'$.
Correction	—	5	30			
Gr. M. T.	19	7	12	$p = 1°\ 27'\ 18''$		
Longitude	10	0	0	$= 87'.3$		
Local M. T.	9	7	12	$\log p$	1.9410	
Sid. T. Gr. noon	23	13	23	$\log \cos t$	$n9.5234$	
Corr. for 19ʰ 7ᵐ	+	3	8	$\log p \cos t$	$n1.4645$	$-p \cos t = +\ 29.1$
$\Theta =$	8	23	43			$\varphi = 31\ \ 39.1$
$a =$	1	5	44			
$t =$	7	17	59			
	$= 109°\ 29'\ 45''$					

Eighth Method.—By Two Altitudes with the elapsed Time between them.

209. This method may be successfully applied at sea, and is the most reliable of all methods, next to that of meridian or circummeridian altitudes. The formulæ fully discussed in Arts. 178 to 183 may be directly applied when the position of the ship has not changed between the observations.

But, since there should be a considerable difference of azimuth between the observations, the change of the ship's position in the interval will generally be sufficiently great to require notice. All that is necessary is to apply a correction to the altitude observed at the first position of the ship, to reduce it to what it would have been if observed at the second position at the same instant.

To obtain this correction, let Z', Fig. 28, be the zenith of the observer at the first observation, S the star at that time; Z his zenith at the second observation, and S' the star at that time. The first observation gives the zenith distance $Z'S$, the second the zenith distance ZS'. Joining the points S and S' with the pole P, it is evident that the hour angle SPS' is obtained from the observed difference of the times of observation precisely as if the observer had been at rest. We have, therefore, only to find ZS in order to have all the data necessary for computing the latitude of Z by the general methods.
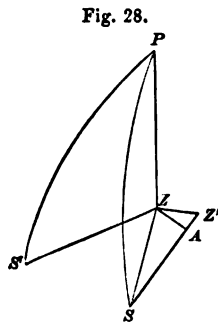
The number of nautical miles run by the ship is the number of minutes in the arc ZZ'; and, since this will always be a suffi-

Fig. 28.

ciently small number, if we draw ZA perpendicular to SZ', we may regard ZAZ' as a plane triangle, and take

$$ZS = Z'S - AZ'$$

or

$$ZS = Z'S - ZZ' \cos ZZ'S \qquad (380)$$

The angle $ZZ'S$ is the difference between the azimuth of the star at the first observation and the course of the ship; and this azimuth is obtained with sufficient accuracy by the compass.*

Employing the zenith distance thus reduced and the other data as observed, the latitude computed by the general method will be that of the second place of observation. In the same manner we can reduce the second zenith distance to the place of the first, and then the latitude of the first place will be found.

210. The problem of finding the latitude from two altitudes is most frequently applied at sea in the case where the sun is the observed body, the observation of the meridian altitude having been lost. The computation is then best carried out by the formulæ (315), (316), (317), (318), employing for δ the mean declination of the sun,—*i.e.* the declination at the middle time between the two observations,—and then applying to the resulting latitude the correction $\Delta\varphi$ found by the formula (323). To save the navigator all consideration of the algebraic signs in computing this correction, it will be sufficient to observe the following rule: 1st. *When the second altitude is the greater*, apply this correction to the computed latitude as a *northing* when the sun is moving towards the *north*, and as a *southing* when the sun is moving towards the *south*; 2d. *When the first altitude is the greater*, apply the correction as a *southing* when the sun is moving towards the *north*, and as a *northing* when the sun is moving towards the *south*.

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* If we wish a more rigorous process, we must consider the spherical triangle $ZZ'S$, in which we have the observed zenith distance $Z'S = (\zeta')$, the required zenith distance $ZS = \zeta$, the distance run by the ship $Z'Z = d$, the difference of the star's azimuth and the ship's course $ZZ'S' = a$, and hence

$$\cos \zeta = \cos \zeta' \cos d + \sin \zeta' \sin d \cos a$$

which developed gives

$$\zeta = \zeta' - d \cos a + \tfrac{1}{2} d^2 \sin 1'' \cot \zeta' \sin^2 a$$

the last term of which expresses the error of the formula given in the text.

If the computer chooses to neglect this correction, he should employ the mean declination only when the middle time is nearer to noon than the time of the greater altitude. In all other cases he should employ the declination for the time of the greater altitude (Art. 183).

211. Douwes's *method of "double altitudes."*[*]—This is a brief method of computing the latitude from two altitudes of the sun, which, though not always accurate, is yet sufficiently so when the interval between the observations is not more than 1^h, and one of them is less than 1^h from the meridian.

Let h and h' be the true altitudes, δ the declination at the middle time, T and T' the chronometer times of the observations, t and t' the hour angles. The elapsed apparent time λ is found from the times T and T' by (322), but it is usually sufficient to take $\lambda = T' - T$. We then have $t' = t + \lambda$; and by the first of (14) we have

$$\sin h = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t$$
$$\sin h' = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos (t + \lambda)$$

The difference of these equations gives

$$\sin h - \sin h' = 2 \cos \varphi \cos \delta \sin (t + \tfrac{1}{2} \lambda) \sin \tfrac{1}{2} \lambda$$

If we put $t_0 =$ the middle time, or

$$t_0 = t + \tfrac{1}{2} \lambda$$

we deduce

$$2 \sin t_0 = \frac{\sin h - \sin h'}{\cos \varphi \cos \delta \sin \tfrac{1}{2} \lambda} \qquad (381)$$

which gives t_0 by employing the supposed latitude for φ in the second member. We then have

$$t = t_0 - \tfrac{1}{2} \lambda$$

and the meridian zenith distance ζ_1 is found from the greater altitude h by the formula (Art. 168)

$$\cos \zeta_1 = \sin h + \cos \varphi \cos \delta (2 \sin^2 \tfrac{1}{2} t)$$

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[*] The method of finding the latitude by two altitudes is commonly called by navigators "the method of double altitudes,"—an obvious misnomer, as double means twice the same.

and finally the latitude by the formula $\varphi = \zeta_1 + \delta$. Since we employ an assumed approximate latitude, we shall have to repeat the process when the computed latitude differs much from the assumed.

This is the form of the method as proposed by DOUWES and adopted in BOWDITCH's Navigator; but the following form is still more simple, as it requires only the table of logarithmic sines. The formula for t_0 may be written thus:

$$\sin t_0 = \frac{\cos \tfrac{1}{2}(h + h') \sin \tfrac{1}{2}(h - h')}{\cos \varphi \cos \delta \sin \tfrac{1}{2}\lambda}$$

then, as before,

$$t = t_0 - \tfrac{1}{2}\lambda$$

and the reduction of h to the meridian altitude h_1 is found by (283),

$$\sin \tfrac{1}{2}(h_1 - h) = \frac{\cos \varphi \cos \delta \sin^2 \tfrac{1}{2}t}{\cos \tfrac{1}{2}(h_1 + h)}$$

Adding $h_1 - h$ to h, we have the meridian altitude, from which the latitude is deduced in the usual manner. If the greater altitude is within the limits of circummeridian altitudes, it will of course be reduced by (284).

The chief objection to this method is that the computation must be repeated when the assumed latitude is much in error. It can also be shown that unless the observations are taken as near to the meridian as we have above supposed, the computed value of the latitude may in certain peculiar cases be more in error than the assumed value, so that successively computed values will more and more diverge from the truth. The methods referred to in the preceding articles are, therefore, generally to be preferred.

212. The latitude may also be found from two altitudes by the simple method proposed by Captain SUMNER, for which see Chapter VIII.

on the time at that place be determined at any instant. The time reckoned at A at this last instant is also known from the correction and rate first found, *provided the rate has not changed in transportation;* and hence the difference of times at the same absolute instant, and consequently the difference of longitude, are found.

Let

$\Delta T, \delta T =$ the correction and rate determined at A at the time T, by the chronometer,

$\Delta T' =$ the correction determined at B at the time $T' = T + t$, t being the interval by the chronometer;

then, at the instant $T + t$ the true time

at A is $T + t + \Delta T + t . \delta T$
" B $T + t + \Delta T'$

and hence the difference of longitude is

$$L = \Delta T + t . \delta T - \Delta T' \tag{383}$$

Thus, the longitude is expressed as the difference of the two chronometer corrections at the two places; and the absolute indications of the chronometer do not enter, except so far as they may be required in determining the interval with which the accumulated rate is computed. In this expression δT is the rate *in a unit of the chronometer* (an hour, or a day, solar or sidereal), and $T' - T$ must be expressed in that unit.

EXAMPLE.—At Greenwich, May 5, mean noon, a mean time chronometer marks $23^h 49^m 42^s.75$, and its rate in 24 chronometer hours has been found to be *gaining* $2^s.671$. At Cambridge, Mass., May 17, mean noon, the same chronometer marks $4^h 34^m 47^s.28$; what is the longitude of Cambridge?

We have

$T = $ May   4, $23^h 49^m 42^s.75$ $\Delta T = + 0^h 10^m 17^s.24$ $\delta T = - 2^s.671$
$T + t = $ "   17,   4   34   47 .23
$\quad\quad t = $ 12^d $4^h 45^m$ $4^s.53 = 12^d.198$

Hence

$$\Delta T + t . \delta T = + 0^h \;\; 9^m \; 44^s.67$$
$$\Delta T' = - 4 \;\; 34 \;\; 47 .28$$
$$\overline{\quad\quad L = + 4 \;\; 44 \;\; 31 .95\quad}$$

Note.—It is proper to distinguish whether the given rate is the rate in a chrono-meter unit or in a true unit of time; although the difference will not be appreciable unless the rate is unusually great. If the rate is 20ˢ in 24ʰ by the chronometer, it will be 20ˢ ± 0ˢ.005 in 24ʰ of solar time.

215. When the chronometer is carried from point to point without stopping to rate it at each, it is convenient to prepare a table of its correction for noon of each day at the first station, from which the correction for the time of any observation at a transient station may be found by simple interpolation.

After reaching the last station, it is proper to re-determine the rate, which will seldom agree precisely with that found at the first. In the absence of any other data affecting the rate, we may assume that it has changed uniformly during the whole time. It is convenient to compute the longitudes first upon the supposition of a constant rate, and then to correct them for the variation of rate, as follows. Let

$\Delta T, \delta T =$ the correction and rate at the time T, found at the first station,

$\delta' T =$ the rate found at the last station at the time $T + n$,

and put

$$x = \frac{\delta' T - \delta T}{n} \tag{384}$$

then x is the increase of rate in a unit of time. If an observation at an intermediate station is taken at the time $T + t$, we must compute the accumulated rate for the interval t, which is effected by multiplying the *mean rate* during this interval by the interval. But, upon the supposition of a uniform increase, the mean rate from the time T to the time $T + t$ is the rate at the middle instant $T + \frac{1}{2} t$, and this rate is $\delta T + \frac{1}{2} t x$. Hence the chronometer correction on the time at the first station at the instant $T + t$ of the supposed observation is

$$\Delta T + t \left(\delta T + \frac{1}{2} t x \right) = \Delta T + t . \delta T + \frac{1}{2} t^2 x \tag{385}$$

A longitude assigned to an intermediate station at the time $T + t$, by employing the original rate δT, will therefore require the correction $+ \frac{1}{2} t^2 x$, observing always the algebraic signs of x and the longitude.

If a number m of chronometers have been employed, and each determination of a longitude is the mean of the m values which they have severally given, the longitude assigned upon the supposition of constant rates is to be corrected by the quantity

$$\frac{t^2}{2} \times \frac{x_1 + x_2 + x_3 + \&c. + x_m}{m}$$

in which x_1, x_2, &c. are the increments of the rates of the several chronometers in a unit of time. If then we put

     $s =$ the sum of all the total increments during the whole interval n, or the sum of the values of $\delta'T - \delta T$ for the several chronometers,

$$q = \frac{s}{2\,mn}$$

we shall have

     Correction of a longitude at a time $T + t = t^2.q$ (386)

EXAMPLE.*—In a voyage between La Guayra and Carthagena, calling on the way at Porto Cabello and Curaçoa, the following observations having been made, the relative longitudes are required.

By observations at La Guayra on May 22 and 28, the corrections and rates of chronometers F, M, and P at the mean epoch May $24^d.885$ were as follows:

	ΔT	δT
Chron. F.	$- 4^h 33^m \ 7^s.80$	$+ 0^s.77$
M.	$- 4 \ \ 0 \ \ 17.40$	$- 4.54$
P.	$- 5 \ \ 9 \ \ 43.70$	$- 1.47$

On arrival at Porto Cabello, the corrections on the mean time at that place on June $5^d.870$ were ascertained to be—

	$\Delta'T$
F.	$- 4^h 37^m 15^s.80$
M.	$- 4 \ \ 5 \ \ 31.28$
P.	$- 5 \ 14 \ 13.38$

At Curaçoa the corrections on June $12^d.890$ were—

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* SHADWELL, Notes on the Management of Chronometers, p. 111.

Now, to correct these results for the changes in the rates of the chronometers, we have, in the interval $n = 33.115$,

	$\delta'T - \delta T$
F.	$+ 0^s.08$
M.	$- 1.36$
P.	$+ 1.77$
$s =$	$+ 0.49$

and, consequently,

$$q = \frac{+ 0^s.49}{2 \times 3 \times 33.115} = + 0^s.002466$$

Applying the correction $t^2 q$ to the several results, the true differences of longitude from La Guayra are found as follows:

	Approx. diff. long.	$t^2.q$	Corrected diff. long.
P. Cabello	$+ 4^m 16^s.25$	$+ 0^s.35$	$+ 4^m 16^s.60$
Curaçoa	$+ 8 \ 3.16$	$+ 0.89$	$+ 8 \ 4.05$
Carthagena	$+ 34 \ 34.51$	$+ 2.70$	$+ 34 \ 37.21$

But it is usually preferable to carry out the result by each chronometer separately, in order to judge of the weight to be attached to the final mean by the agreement of the several individual values. For this purpose we have here, by the formula (384), for $n = 33.115$,

	$\frac{1}{2} x$
F.	$+ 0.00121$
M.	$- 0.02054$
P.	$+ 0.02673$

and hence the correction $\frac{1}{2} t^2 . x$ is, for the several cases, as follows:

	P. Cabello.	Curaçoa.	Carthagena.
F.	$+ 0^s.17$	$+ 0^s.44$	$+ 1^s.32$
M.	$- 2.95$	$- 7.41$	$- 22.52$
P.	$+ 3.84$	$+ 9.65$	$+ 29.31$

Applying these corrections severally to the above approximate results, we have, for the differences of longitude from La Guayra,

	P. Cabello.	Curaçon.	Carthagena.
F.	$+ 4^m 17^s.40$	$+ 8^m 6^s.47$	$+ 34^m 42^s.57$
M.	16.52	4.44	37.72
P.	15.90	1.25	31.35
Means	$+ 4 \ 16.61$	$+ 8 \ 4.05$	$+ 34 \ 37.21$

agreeing precisely with the corrected means found above.

If the chronometers have been exposed to considerable changes of temperature, the proper correction may be introduced by the method of Art. 223.

216. *Chronometric expeditions between two points.*—Where a difference of longitude is to be determined with the greatest possible precision, a large number of chronometers are transported back and forth between the extreme points. There are two classes of errors of chronometers which are to be eliminated: 1st, the *accidental* errors, or variations of rate which follow no law, and may be either positive or negative; 2d, the *constant* errors, or variations of rate which, for any given chronometer, appear with the same sign and of the same amount when the chronometer is transported from place to place; in other words, a constant acceleration, or a constant retardation, as compared with the rates found when the chronometer is at rest. The accidental errors are eliminated in a great degree by employing a large number of chronometers, the probability being that such errors will have different signs for different chronometers. The constant errors cannot be determined by comparing the rates at the two extreme points, since these rates are found only when the chronometer is at rest; but if the chronometers are transported in both directions, from east to west and from west to east, a constant error in their *travelling* rates will affect the difference of longitude with opposite signs in the two journeys, and will disappear when the mean is taken. These considerations have given rise to extensive expeditions, of which probably the most thoroughly executed was that carried out by STRUVE, in 1843, between Pulkova and Altona.* In this expedition sixty-eight chronometers were transported eight times from Pulkova to Altona and back, making sixteen voyages in all, giving the difference of longitude between the centre of the Pulkova Observatory and the Altona Observatory $1^h 21^m 32^s.527$, with a probable error of only $0^s.039$.

Chronometric expeditions between Liverpool (England) and

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* *Expédition chronométrique exécutée par ordre de Sa Majesté L'Empereur Nicolas I. pour la détermination de la longitude géographique relative de l'observatoire centrale de Russie.* St. Petersburg, 1844.

For an account of the carefully executed expedition under Professor AIRY to determine the longitude of Valentia in Ireland, see the Appendix to the Greenwich Observations of 1845.

Cambridge (U. S.) were instituted in the years 1849, '50, '51, and '55 by the U. S. Coast Survey, under the superintendence of Professor A. D. Bache. The results of the expeditions of 1849, '50, and '51, discussed by Mr. G. P. Bond,* proved the necessity of introducing a correction for the temperature to which the chronometers were exposed during the voyages, and particular attention was therefore paid to this point in the expedition of 1855, the details of which were arranged by Mr. W. C. Bond. The results of six voyages,—three in each direction,—according to the discussion of Mr. G. P. Bond,† were as follows:

<div style="margin-left:auto; text-align:right;">Longitude.</div>

Voyages from Liverpool to Cambridge	$4^h\ 32^m\ 31^s.92$
" " Cambridge to Liverpool	4 32 31 .75
Mean	4 32 31 .84

with a probable error of $0^s.19$. In this expedition fifty chronometers were used. The greater probable error of the result, as compared with Struve's, is sufficiently explained by the greater length of the voyages and their smaller number.

217. The following is essentially Struve's method of conducting the expeditions and discussing the results.

Before embarking the chronometers at the first station (A), they are carefully compared with a standard clock the correction of which on the time at that station has been obtained with the greatest precision by transits of well-determined stars. (See Vol. II., "Transit Instrument.") Upon their arrival at the second station (B), they are compared with the standard clock at that station.‡ From these two comparisons the chronometer corrections at the two stations become known, and, if the rates are known, a value of the longitude is found by each chronometer by (383). But here it is to be observed that the rate of a chronometer is rarely the same when in motion as when at rest. It is necessary, therefore, to find its *travelling rate* (or *sea rate*, as it is called when the chronometer is transported by sea). This might be effected by finding—*first*, the correction of the chrono-

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* Report of the Superintendent of the U. S. Coast Survey for 1854, Appendix No. 42.

† Report of the Superintendent of the U. S. Coast Survey for 1856, p. 182.

‡ For the method of comparing chronometers and clocks with the greatest precision, see Vol. II.

meter at the station A immediately before starting; *secondly*, its correction at B immediately upon its arrival there; and *thirdly*, having, without any delay at B, returned directly to A, finding again its correction there immediately upon arriving. The difference between the two corrections at A is the whole travelling rate during the elapsed time, and this rate would be used in making the comparison with the correction obtained at B, and in deducing the longitude by (383).

But, since the chronometer cannot generally be *immediately* returned from B, its correction for that station should be found both upon its arrival there and again just before leaving, and the travelling rate inferred only from the time the instrument is in motion. For this purpose, let us suppose that we have found

| at the times | t, | t', | t'', | t''', |
| the chron. corrections | a, | b, | b', | a', |

the correction a at the station A before leaving; b upon arriving at B; b' before leaving B; and a' upon the return to A. The times t, t', t'', t''', being all reckoned at the same meridian, if we now put

$m =$ the mean travelling rate of the chronometer in a unit of time,

$\lambda =$ the longitude of B west of A,

we shall have, upon the supposition that the mean travelling rate is the same for both the east and west voyages,

$$\lambda = a + m (t' - t) - b$$
$$\lambda = a' - m (t''' - t'') - b'$$

From these two equations the two unknown quantities m and λ become known. Putting

$$\tau = t' - t \qquad \tau'' = t''' - t''$$

we find, first,

$$m = \frac{(a' - a) - (b' - b)}{\tau + \tau''} \qquad (387)$$

in which the numerator evidently expresses the whole travelling rate, and the denominator the whole travelling time. Then, putting

$$(a) = a + m\tau$$

we have

$$\lambda = (a) - b$$

$$\Bigg\} (388)$$

in which (a) is the interpolated value of the chronometer correction on the time at A, for the same absolute instant t' to which the correction b on the time at B corresponds.

EXAMPLE.—In the first two voyages of STRUVE'S expedition between Pulkova and Altona in 1843, the corrections of the chronometer "Hauth 31" were found, by comparison with the standard clocks at the two stations, as below. The dates are all in Pulkova time, as shown by one of the chronometers employed in the comparison:

At Pulkova (A), $t = $ May 19, $21^h.54$ $a = + 0^h \ 6^m \ 38^s.10$
" Altona (B), $t' = $ "   24, 22 .66 $b = - 1 \ 14 \ 39 .92$
" Altona (B), $t'' = $ "   26, 10 .72 $b' = - 1 \ 14 \ 36 .77$
" Pulkova (A), $t''' = $ "   31, 0 .00 $a' = + 0 \ \ 7 \ \ 9 .58$

Hence

$$\tau = 5^d \ \ 1^h.12 = 5^d.047, \qquad a' - a = + 31^s.48$$
$$\tau'' = 4 \ \ 13 \ .28 = 4 \ .553, \qquad b' - b = + \ \ 3 .15$$

$$m = \frac{31^s.48 - 3^s.15}{5.047 + 4.553} = \frac{28^s.33}{9.6} = + 2^s.951$$

$$
\begin{aligned}
a &= + 0^h \ \ 6^m \ 38^s.10 \\
m\tau &= \qquad \quad + \ \ 14 .89 \\
\hline
(a) &= + 0 \ \ \ 6 \ \ 52 .99 \\
b &= - 1 \ \ 14 \ \ 39 .92 \\
\hline
\lambda = (a) - b &= + 1 \ \ 21 \ \ 32 .91
\end{aligned}
$$

218. In the above, the rate of the chronometer is assumed to be constant, and the problem is treated as one of simple interpolation. But most chronometers exhibit more or less acceleration or retardation in successive voyages, and a strict interpolation requires that we should have regard to second differences. If we always start from the station A, as in the above example, using only simple interpolation, we commit a small error, which always affects the longitude in the same way so long as the variation of the chronometer's rate preserves the same sign. But if we commence the next computation with the station B,

$$(b) = b + \left(\frac{b' - b}{\tau' + \tau''}\right)\tau'$$

$$= a + \mu(\tau + \tau') + \beta(\tau^2 + 2\tau\tau' + \tau'^2 + \tau'\tau'') - \lambda$$

and we find the erroneous longitude

$$\lambda'' = a' - (b) = \lambda - \beta\tau'\tau''$$

Hence the error by simple interpolation, commencing with the station B, is $d\lambda'' = -\beta\tau'\tau''$; and the error in the mean of the two longitudes is

$$\tfrac{1}{2}(d\lambda' + d\lambda'') = \tfrac{1}{2}\beta\tau'(\tau - \tau'')$$

an error which disappears altogether when the intervals τ and τ'' are equal. Since the voyages are of very nearly equal duration, it follows that by computing the longitude, as proposed by STRUVE, commencing alternately at the two stations, the final result will be free from the effect of any regular acceleration or retardation of the chronometers.

EXAMPLE.—From the "Expédition Chronométrique" we take the following values for the chronometer "Hauth 31," being the combination next following after that given in the example of the preceding article, commencing now with the station B, or Altona:

At Altona $(B), t = $ May 26, $10^h.72$ $b = -1^h\ 14^m\ 36^s.77$
" Pulkova $(A), t' = $ " 31, 0 .00 $a = +0\quad 7\quad 9 .58$
" Pulkova $(A), t'' = $ June 3, 5 .62 $a' = +0\quad 7\quad 19 .36$
" Altona $(B), t''' = $ " 7, 20 .52 $b' = -1\ 14\quad 0 .35$

Here

$$\tau = 4^d\ 13^h.28 = 4^d.553 \qquad b' - b = +36^s.42$$
$$\tau'' = 4\ \ 14 .90 = 4 .621 \qquad a' - a = +\ 9 .78$$

$$m = \frac{36^s.42 - 9^s.78}{4.553 + 4.621} = \frac{26^s.64}{9.174} = +2^s.904$$

$$
\begin{aligned}
b &= -1^h\ 14^m\ 36^s.77 \\
m\tau &= \underline{\quad +\quad 13 .22} \\
(b) &= -1\ 14\ 23 .55 \\
a &= +0\quad 7\quad 9 .58 \\
\lambda = a - (b) &= +1\ 21\ 33 .13
\end{aligned}
$$

The mean of this result and that of Art. 217 is $\lambda = 1^h\ 21^m\ 33^s.02$.

219. *Relative weight of the longitudes determined in different voyages by the same chronometer.*—From the above it appears that the problem of finding the longitude by chronometers is one of interpolation. If the irregularities of the chronometer are regarded as accidental, the mean error of an interpolated value of the correction may be expressed by the formula*

$$\mp \varepsilon \sqrt{\frac{\tau\tau'}{\tau + \tau'}}$$

where τ and τ' have the same signification as in the preceding article, and ε is the mean (accidental) error in a unit of time. The weight of such an interpolated value of the correction, and, therefore, also the weight of a value of the longitude deduced from it, is inversely proportional to the square of this error, and may, therefore, be expressed under the form

$$p = k \cdot \frac{\tau + \tau'}{\tau\tau'}$$

where k is a constant arbitrarily taken for the whole expedition, so as to give p convenient values, since it is only the *relative* weights of the different voyages which are in question.

But if the chronometer variations are no longer accidental, but follow some law though unknown, a special investigation may serve to give empirically a more suitable expression of the weight than the above. Thus, according to STRUVE's investigations in the case of certain clocks, the weight of an interpolated value of the correction for these clocks could be well expressed by the formula†

$$p = k \left(\frac{\tau + \tau'}{\tau\tau'} \right)^2$$

But even this expression he found could not be generally applied; and he finally adopted the following form for the chronometric expedition:

$$p = \frac{K}{T\sqrt{\tau\tau''}} \tag{389}$$

in which T is the duration of an entire voyage, including the

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* See Vol. II., "Chronometer."
† *Expédition Chron.*, p. 102.

chronometers are inversely proportional to their mean errors. The weight P of a longitude L will, therefore, be expressed generally by

$$P = \frac{k}{\varepsilon\varepsilon}$$

in which k is arbitrary. For simplicity, we may assume $k = 1$, and then by the above value of ε we shall have

$$P = \frac{(n-1)[p]}{[pvv]} \tag{392}$$

If, then, L', L'', L'''..... are the values found by the several chronometers by (390), P', P'', P'''..... their weights by (392), the most probable final value of the longitude is

$$L_0 = \frac{P'L' + P''L'' + P'''L''' + \cdots}{P' + P'' + P''' + \cdots} \tag{393}$$

Then, putting

$$L' - L_0 = V', \qquad L'' - L_0 = V'', \qquad L''' - L_0 = V''' \quad \&c.$$
$$N = \text{the number of values of } L,$$
$$E = \text{the mean error of } L_0,$$
$$R = \text{the probable error of } L_0,$$

we have

$$E = \sqrt{\frac{[PVV]}{(N-1)[P]}} \qquad R = 0.6745\,E \tag{394}$$

222. I propose to illustrate the preceding formulæ by applying them to two chronometers of STRUVE's expedition, namely, "Dent 1774" and "Hauth 31." In the following table the longitudes found by beginning at Pulkova are marked P, those found by beginning at Altona are marked A, and the numeral accent denotes the number of the voyage. The weights p in the second column are as given by STRUVE, who computed them by the formula (389), taking $K = 34560$ (the intervals T, τ, τ'' being in hours), which is a convenient value, as it makes the weight of a voyage of nearly mean duration equal to unity; namely, for $T = 288^h$, $\tau = \tau' = 120^h$. If we express T, τ, τ'', in days, we take

$$K = \frac{34560}{(24)^2} = 60$$

and we shall have STRUVE's values of p by the formula

$$p = \frac{60}{T\sqrt{\tau\tau''}} \tag{395}$$

Thus, for the first voyage, we have, from the data in the example of Art. 217,

$$T = t''' - t = 11^d\ 2^h.46 = 11^d.103$$
$$\tau = 5^s.047 \qquad\qquad \tau'' = 4^s.553$$

whence, by (395),

$$p = \frac{60}{11.103\ \sqrt{(5.047 \times 4.553)}} = 1.13$$

The values of L' and L'' are found by (390). In applying this formula, it is not necessary to multiply the entire longitudes by their weights, but only those figures which differ in the several values. Thus, by "Dent 1774" we have

$$L' = 1^h\ 21^m\ 30^s + \frac{2^s.51 \times 1.10 + 2^s.83 \times 1.02 + 0^s.09 \times 1.14 + \&c.}{1.10 + 1.02 + 1.14 + \&c.}$$
$$= 1^h\ 21^m\ 30^s + 2^s.46$$

	Weight. p	Longitudes by Chronometer Dent 1774.	v	pvv	Longitudes by Chronometer Hauth 31.	v	pvv
P^i	1.13				$1^h\ 21^m\ 32^s.91$	$+ 0^s.30$	0.102
A^i	1.06				83 .18	$+ 0.52$	0.287
P^ii	1.10	$1^h\ 21^m\ 32^s.51$	$+ 0^s.05$	0.003	33 .36	$+ 0.75$	0.619
A^ii	1.02	32 .83	$+ 0.87$	0.140	83 .12	$+ 0.51$	0.265
P^iii	1.14	32 .09	$- 0.37$	0.156	82 .55	$- 0.06$	0.004
A^iii	1.05	32 .25	$- 0.21$	0.046	31 .56	$- 1.05$	1.158
P^iv	1.19	31 .69	$- 0.77$	0.706	82 .70	$+ 0.09$	0.010
A^iv	0.96	32 .77	$+ 0.31$	0.092	34 .16	$+ 1.55$	2.306
P^v	1.09	32 .79	$+ 0.33$	0.119	82 .23	$- 0.38$	0.157
A^v	0.80	32 .54	$+ 0.08$	0.005	31 .65	$- 0.96$	0.737
P^vi	1.00	32 .94	$+ 0.48$	0.230	83 .38	$+ 0.77$	0.593
A^vi	1.10	31 .93	$- 0.53$	0.309	81 .97	$- 0.64$	0.451
P^vii	1.20	32 .34	$- 0.12$	0.017	83 .16	$+ 0.55$	0.363
A^vii	1.09	32 .95	$+ 0.49$	0.262	31 .78	$- 0.83$	0.751
P^viii	0.76	31 .86	$- 0.60$	0.274	80 .92	$- 1.69$	2.171
A^viii	0.41	83 .77	$+ 1.31$	0.704			

$$L' = 1^h\ 21^m\ 32^s.46 \qquad [pvv] = 8.063$$
$$n = 14 \qquad [p] = 13.91$$

$$P' = \frac{13 \times 13.91}{8.063} = 59.04$$

$$r' = \pm \frac{.6745}{\sqrt{P'}} = \pm 0^s.09$$

$$L'' = 1^h\ 21^m\ 32^s.61 \qquad [pvv] = 9.974$$
$$n = 15 \qquad [p] = 15.69$$

$$P'' = \frac{14 \times 15.69}{9.974} = 22.02$$

$$r'' = \pm \frac{.6745}{\sqrt{P''}} \pm 0^s.14$$

Combining these two results, we have, by (393),

$$L_0 = 1^\text{h} 21^\text{m} 32^\text{s} + \frac{0^\text{s}.46 \times 59 + 0^\text{s}.61 \times 22}{59 + 22} = 1^\text{h} 21^\text{m} 32^\text{s}.501$$

with the probable error, by (394),

$$R = \pm 0^\text{s}.067$$

This agrees very nearly with the final result from the sixty-eight chronometers.

223. In the preceding method, the sea rate is inferred from two comparisons of the chronometer made at the same place before and after the voyages to and from the second place; and the correction of the chronometer on the time of the first place at the instant when it is compared with the time of the second place is interpolated upon the theory that the rate has changed uniformly. This theory is insufficient when the temperature to which the chronometer is exposed is not constant during the two voyages, or nearly so. I shall, therefore, add the method of introducing the correction for temperature in cases where circumstances may seem to require it.

According to the experience of M. LIEUSSON, the rate m of a chronometer at a given temperature ϑ may be expressed by the formula (see Vol. II., "Chronometer")

$$m = m_0 + k(\vartheta - \vartheta_0)^2 - k't \tag{396}$$

in which ϑ_0 is the temperature for which the balance is compensated, m_0 the rate determined at that temperature at the epoch $t = 0$, t being the time from this epoch for which the rate m is required, k the constant coefficient of temperature, and k' that of acceleration of the chronometer resulting from thickening of the oil or other gradual changes which are supposed to be proportional to the time.

It is evident that, since every change of temperature produces an increase of m, the term $k(\vartheta - \vartheta_0)^2$ will not disappear even when the *mean* value of ϑ is the same as ϑ_0. It is necessary, therefore, to determine the sum of the effects of all the changes. Let us, therefore, determine the accumulated rate for a given period of time τ. Let m_0 be the rate at the middle of this period, in which case we have in the formula $t = 0$. A strict theory requires that

we should know the temperature at every instant; but, in default of this, let us assume that the period τ is divided into sufficiently small intervals, and that the temperature is observed in each. Let us suppose n equal intervals whose sum is τ, and denote the observed values of ϑ by $\vartheta^{(1)}$, $\vartheta^{(2)}$, $\vartheta^{(3)}$.... $\vartheta^{(n)}$. The rate

in the 1st interval is $[m_0 + k(\vartheta^{(1)} - \vartheta_0)^2] \times \dfrac{\tau}{n}$

       "   2d   " $[m_0 + k(\vartheta^{(2)} - \vartheta_0)^2] \times \dfrac{\tau}{n}$

&c.   &c.

in the nth interval is $[m_0 + k(\vartheta^{(n)} - \vartheta_0)^2] \times \dfrac{\tau}{n}$

and the accumulated rate in the time τ is the sum of these quantities,

$$= m_0\tau + k\, \Sigma_n (\vartheta - \vartheta_0)^2\, \frac{\tau}{n}$$

where $\Sigma_n (\vartheta - \vartheta_0)^2$ denotes the sum of the n values of $(\vartheta - \vartheta_0)^2$. To make this expression exact, we should have an infinite number of infinitesimal intervals, or we must put $\dfrac{\tau}{n} = d\tau$, and substitute the integral sign $\int$ for the summation symbol Σ: thus, the exact expression for the whole rate in the time τ is

$$m_0\tau + k \int_0^\tau (\vartheta - \vartheta_0)^2 \, d\tau \qquad (397)$$

This integral cannot be found in general terms, since ϑ cannot be expressed as a function of τ; but we can obtain an approximate expression for it, as follows. Let ϑ_1 be the mean of all the observed values of ϑ; then we have

$$\Sigma_n (\vartheta - \vartheta_0)^2 = \Sigma_n [(\vartheta_1 - \vartheta_0) + (\vartheta - \vartheta_1)]^2$$
$$= \Sigma_n (\vartheta_1 - \vartheta_0)^2 + \Sigma_n 2(\vartheta_1 - \vartheta_0)(\vartheta - \vartheta_1) + \Sigma_n (\vartheta - \vartheta_1)^2$$

in which $\vartheta_1 - \vartheta_0$ is constant, and, therefore, for n values we have $\Sigma_n (\vartheta_1 - \vartheta_0)^2 = n(\vartheta_1 - \vartheta_0)^2$. Moreover, since ϑ_1 is the mean of all the values of ϑ, we have $\Sigma_n (\vartheta - \vartheta_1) = 0$, and, consequently, also $\Sigma_n 2(\vartheta_1 - \vartheta_0)(\vartheta - \vartheta_1) = 2(\vartheta_1 - \vartheta_0)\, \Sigma_n (\vartheta - \vartheta_1) = 0$; and the above expression becomes

$$\Sigma_n (\vartheta - \vartheta_0)^2 = n(\vartheta_1 - \vartheta_0)^2 + \Sigma_n (\vartheta - \vartheta_1)^2$$

Hence, also,

$$\Sigma_n (\vartheta - \vartheta_0)^2 \frac{\tau}{n} = \tau (\vartheta_1 - \vartheta_0)^2 + \Sigma_n (\vartheta - \vartheta_1)^2 \frac{\tau}{n}$$

or, for an infinite value of n,

$$\int_0^\tau (\vartheta - \vartheta_0)^2 d\tau = \tau (\vartheta_1 - \vartheta_0)^2 + \int_0^\tau (\vartheta - \vartheta_1)^2 d\tau$$

Thus, the required integral depends upon the integral $\int_0^\tau (\vartheta - \vartheta_1)^2 d\tau$, which may be approximately found from the observed values of ϑ by the theory of least squares. For, if we treat the values of $\vartheta - \vartheta_1$ as the errors of the observed values of ϑ, and denote the mean error (according to the received acceptation of that term in the method of least squares) by e, we have

$$e^2 = \frac{\Sigma_n (\vartheta - \vartheta_1)^2}{n - 1} \tag{398}$$

in which n is the actual number of observed values of ϑ. If we assume that a more extended series of values, or indeed an infinite series, would exhibit the same mean error (which will be the more nearly true the greater the number n), we assume the general relation

$$\Sigma_N (\vartheta - \vartheta_1)^2 = (N - 1) e^2$$

in which N is any number. Hence, also,

$$\Sigma_N (\vartheta - \vartheta_1)^2 \frac{\tau}{N} = \tau e^2 \frac{N - 1}{N}$$

and, making N infinite,

$$\int_0^\tau (\vartheta - \vartheta_1)^2 d\tau = \tau e^2 \tag{399}$$

Substituting this value, the formula (397) becomes

$$m_0 \tau + k \tau (\vartheta_1 - \vartheta_0)^2 + k \tau e^2$$

or
$$[m_0 + k (\vartheta_1 - \vartheta_0)^2 + k e^2] \tau \tag{400}$$

from which it appears that $m_0 + k (\vartheta_1 - \vartheta_0)^2 + k e^2$ is the mean rate in a unit of time for the interval τ, m_0 being the rate at the middle of the interval for a temperature $\vartheta = \vartheta_0$. For any subsequent interval τ', we must, according to (396), replace m_0 by $m_0 - k't$, t being the interval from the middle of τ to the middle of τ'.

Now, let us suppose that the chronometer correction is obtained by astronomical observations at the station A, at the times T_1 and T_2, before starting upon the voyage, and again after reaching the station B, at the times T_3 and T_4, these times being all reckoned at the same meridian. Let a_1, a_2, a_3, a_4, be the observed corrections, and put

$$T_2 - T_1 = \tau, \qquad T_3 - T_2 = \tau', \qquad T_4 - T_3 = \tau''$$

so that τ and τ'' are the shore intervals and τ' the sea interval. Let the adopted epoch of the rate m_0 be the middle of the sea interval τ'; then, by (400), with the correction $k't$, the accumulated rates in the three intervals are

$$
\left.
\begin{aligned}
a_2 - a_1 &= \left[m_0 + k' \left(\frac{\tau + \tau'}{2} \right) + k(\vartheta_1 - \vartheta_0)^2 + ke^2 \right] \tau \\
\lambda + a_3 - a_2 &= \left[m_0 \qquad\qquad\quad + k(\vartheta_1' - \vartheta_0)^2 + ke'^2 \right] \tau' \\
a_4 - a_3 &= \left[m_0 - k' \left(\frac{\tau'' + \tau'}{2} \right) + k(\vartheta_1'' - \vartheta_0)^2 + ke''^2 \right] \tau''
\end{aligned}
\right\}
\quad (401)
$$

in which $\vartheta_1, \vartheta_1', \vartheta_1''$ are the mean temperatures in the intervals τ, τ', τ'', and e, e', e'' are found by the formula (398). These three equations determine the three unknown quantities $m_0, k',$ and λ. If we put

$$f = \frac{a_2 - a_1}{\tau} - k(\vartheta_1 - \vartheta_0)^2 - ke^2$$

$$f'' = \frac{a_4 - a_3}{\tau''} - k(\vartheta_1'' - \vartheta_0)^2 - ke''^2$$

we have, from the first and third equations,

$$k' = \frac{f - f''}{\tau' + \tfrac{1}{2}(\tau + \tau'')}$$

$$m_0 = \frac{f + f''}{2} + \tfrac{1}{4} k' (\tau'' - \tau)$$

which substituted in the second equation gives λ. If, however, we prefer to compute the approximate longitude without considering the temperatures, and afterwards to correct for temperature, we shall have

second signal the observers at B and C the times b' and c; at the third signal the observers at C and D the times c' and d; it is evident that the time at A when the third signal is made is $a + (b' - b) + (c' - c)$, at which instant the time at D is d: hence the difference of longitude of A and D is

$$\lambda = a + (b' - b) + (c' - c) - d \qquad (403)$$

and so on for any number of intermediate stations. It is required of the intermediate chronometers only that they should give correctly the differences $b' - b$, $c' - c$, for which purpose only their rates must be accurately known. The daily rates are obtained by a comparison of the instants of the signals on successive days. Small errors in the rates will be eliminated by making the signals both from west to east and from east to west, and taking the mean of the results.

The intervals given by the intermediate chronometers should, of course, be reduced to sidereal intervals, if the clocks at the extreme stations are regulated to sidereal time.

EXAMPLE.—From the *Description Géométrique de la France* (PUISSANT). On the 25th of August, 1824, signals were observed between *Paris* and *Strasburg* as follows:

Paris.	Intermediate Stations.		Strasburg.
A	B	C	D
$19^h 6^m 20^s.3$	$8^h 49^m 48^s.2$		
	$8\ 54\ 10.8$	$9^h 16^m\ 0^s.2$	
		$9\ 30\ 37.8$	$19^h 46^m 51^s.4$

The correction of the Paris clock on Paris sidereal time was $-36^s.2$; that of the Strasburg clock on Strasburg sidereal time was $-27^s.7$. The chronometers at B and C were regulated to mean time, and their daily rates were so small as not to be sensible in the short intervals which occurred.

We have

$$
\begin{aligned}
b' - b &= 4^m\ 22^s.6 \\
c' - c &= 14\ \ 37.6 \\
\hline
\text{Mean interval} &= 19\ \ \ \,0.2 \\
\text{Red. to sid. int.} &= + 3.1 \\
\hline
\text{Sid. interval} &= 19\ \ \ \,3.3
\end{aligned}
$$

Paris clock	19^h 6^m $20^s.3$	Strasburg clock	19^h 46^m $51^s.4$
Correction	$-$ 36 .2	Correction	$-$ 27 .7
Paris sid. time	19 5 44 .1	Strasburg sid. time	19 46 23 .7
Sid. interval	$+$ 19 3 .3		
Paris sid. time of the } last signal	19 24 47 .4		
Strasburg do.	19 46 23 .7		
	$\lambda =$ 0^h 21^m $36^s.3$		

In the survey of the boundary between the United States and Mexico, Major W. H. EMORY, in 1852, employed flashes of gunpowder as signals in determining the diff. of long. of Frontera and San Elciario.*

The signals may be given by the *heliotrope* of GAUSS, by which an image of the sun is reflected constantly in a given direction towards the distant observer. Either the sudden eclipse of the light, or its reappearance, may be taken as the signal; the eclipse is usually preferred.

Among the methods by terrestrial signals may be included that in which the signal is given by means of an electro-telegraphic wire connecting the two stations; but this important and exceedingly accurate method will be separately considered below.

225. *Celestial Signals.*—Certain celestial phenomena which are visible *at the same absolute instant* by observers in various parts of the globe, may be used instead of the terrestrial signals of the preceding article: among these we may note—

a. The bursting of a meteor, and the appearance or disappearance of a shooting star.—The difficulty of identifying these objects at remote stations prevents the extended use of this method.

b. The instant of beginning or ending of an eclipse of the moon.—This instant, however, cannot be accurately observed, on account of the imperfect definition of the earth's shadow. A rude approximation to the difference of longitude is all that can be expected by this method.

c. The *eclipses* of Jupiter's satellites by the shadow of that planet.—The Greenwich times of the disappearance of each

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* Proceedings of 8th Meeting of Am. Association, p. 64.

satellite, and of its reappearance, are accurately given in the Ephemeris: so that an observer who has noted one of these phenomena has only to take the difference between this observed local time of its occurrence and the Greenwich time given in the Ephemeris, to have his absolute longitude. With telescopes of different powers, however, the instant of a satellite's disappearance must evidently vary, since the eclipse of the satellite takes place gradually, and the more powerful the telescope the longer will it continue to show the satellite. If the disappearance and reappearance are both observed with the same telescope, the mean of the results obtained will be nearly free from this error. The first satellite is to be preferred, as its eclipses occur more frequently and also more suddenly. Observers who wish to deduce their difference of longitude by these eclipses should use telescopes of the same power, and observe under the same atmospheric conditions, as nearly as possible. But in no case can extreme precision be attained by this method.

d. The *occultations* of Jupiter's satellites by the body of the planet.—The *approximate* Greenwich times of the disappearance behind the disc, and the reappearance of each satellite, are given in the Ephemeris. These predicted times serve only to enable the observers to direct their attention to the phenomenon at the proper moment.

e. The *transits* of the satellites over Jupiter's disc.—The approximate Greenwich times of "ingress" and "egress," or the first and last instants when the satellite appears projected on the planet's disc, are given in the Ephemeris.

f. The *transits of the shadows* of the satellites over Jupiter's disc.—The Greenwich times of "ingress" and "egress" of the shadow are also approximately given in the Ephemeris.

g. Among the celestial signals we may include also eclipses of the sun, or occultations of stars and planets by the moon, or, in general, the arrival of the moon at any given position in the heavens; but, in consequence of the moon's parallax, these eclipses and occultations do not occur at the same absolute instant for all observers, and, in general, the moon's apparent position in the heavens is affected by both parallax and refraction. The methods of employing these phenomena as signals, therefore, involve special computations, and will be hereafter treated of. See the general theory of eclipses, and the method of lunar distances.

THIRD METHOD.—BY THE ELECTRIC TELEGRAPH.

226. It is evident that the clocks at two stations, A and B, may be compared by means of signals communicated through an electro-telegraphic wire which connects the stations. Suppose at a time T by the clock at A, a signal is made which is perceived at B at the time T' by the clock at that station. Let ΔT and $\Delta T'$ be the clock corrections on the times at these stations respectively (both being solar or both sidereal). Let x be the time required by the electric current to pass over the wire; then, A being the more easterly station, we have the difference of longitude λ by the formula

$$\lambda = (T + \Delta T) - (T' + \Delta T') + x = \lambda_1 + x$$

Since x is unknown, we must endeavor to eliminate it. For this purpose, let a signal be made at B at the clock time T'', which is perceived at A at the clock time T'''; then we have

$$\lambda = (T''' + \Delta T''') - (T'' + \Delta T'') - x = \lambda_2 - x$$

In these formulæ λ_1 and λ_2 denote the approximate values of the difference of longitude, found by signals east-west and west-east respectively, when the transmission time x is disregarded; and the true value is

$$\lambda = \tfrac{1}{2} (\lambda_1 + \lambda_2)$$

Such is the simple and obvious application of the telegraph to the determination of longitudes; but the degree of accuracy of the result depends greatly—more than at first appears—upon the manner in which the signals are communicated and received.

Suppose the observer at A taps upon a signal key* at an exact second by his clock, thereby producing an audible click of the armature of the electro-magnet at B. The observer at B may not only determine the nearest second by his clock when he hears this click, but may also estimate the fraction of a second; and it would seem that we ought in this way to be able to determine a longitude within one-tenth of a second. But, before even this degree of accuracy can be secured, we have yet to eliminate, or reduce to a minimum, the following sources of error:

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* See Vol. II., "Chronograph," for the details of the apparatus here alluded to.

server indicates it by another preconcerted signal, the chronographs are stopped, and the record is suitably marked with date, name of the star, and place of observation, to be subsequently identified and read off accurately by a scale. When the star arrives at the meridian of B, the transit is recorded in the same manner upon both chronographs.

Suitable observations having been made by each observer to determine the errors of his transit instrument and the rate of the clock, let us put

T_1 = the mean of the clock times of the eastern transit of the star over all the threads, as read from the chronograph at A,

T_2 = the same, as read from the chronograph at B,

T_1' = the mean of the clock times of the western transit of the star over all the threads, as read from the chronograph at A,

T_2' = the same as read from the chronograph at B,

e, e' = the personal equations of the observers at A and B respectively,

τ, τ' = the corrections of T_1 and T_1' (or of T_2 and T_2') for the state of the transit instruments at A and B, or the respective "reductions to the meridian" (Vol. II., Transit Inst.),

δT = the correction for clock rate in the interval $T_1' - T_1$,

x = the transmission time of the electric current between A and B,

λ = the difference of longitude;

then it is easily seen that we have, from the chronographic records at A,

$$\lambda = T_1' + \delta T + \tau' + e' - x - (T_1 + \tau + e)$$

and from the chronographic records at B,

$$\lambda = T_2' + \delta T + \tau' + e' + x - (T_2 + \tau + e)$$

and the mean of these values is

$$\lambda = [\tfrac{1}{2}(T_1' + T_2') + \tau'] - [\tfrac{1}{2}(T_1 + T_2) + \tau] + \delta T + e' - e \quad (404)$$

which we may briefly express thus:

$$\lambda = \lambda_1 + e' - e.$$

in which

> $\lambda_1 =$ the approximate difference of longitude found by the exchange of star signals, when the personal equations of the observers are neglected.

This equation would be final if $e' - e$, or the relative personal equation of the observers, were known: however, if the observers now exchange stations and repeat the above process, we shall have, provided the relative personal equation is constant,

$$\lambda = \lambda_2 + e - e'$$

in which λ_2 is the approximate difference of longitude found as before; and hence the final value is

$$\lambda = \tfrac{1}{2}(\lambda_1 + \lambda_2)$$

I have not here introduced any consideration of the armature time, because it affects clock signals and star signals in the same manner; and therefore the time read from the chronographic fillet or sheet is the same as if the armature acted instantaneously.* It is necessary, however, that this time should be constant from the first observation at the first station to the last observation at the second, and therefore it is important that no changes should be made in the adjustments of the apparatus during the interval.

As the observer has only to tap the transits of the star over the threads, the latter may be placed very close together. The reticules prepared by Mr. W. WÜRDEMANN for the Coast Survey have generally contained twenty-five threads, in groups or "tallies" of five, the equatorial intervals between the threads, of a group being 2ˢ.5, and those between the groups 5ˢ; with an additional thread on each side at the distance of 10ˢ for use in observations by "eye and ear." Except when clouds intervene and render it necessary to take whatever threads may be available, only the three middle tallies, or fifteen threads, are used. The use of more has been found to add less to the accuracy of a

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* Dr. B. A. GOULD thinks that the armature time varies with the strength of the battery and the distance (and consequent weakness) of the signal; being thus liable to be confounded with the transmission time. The effect upon the difference of longitude will be inappreciable if the batteries are maintained at nearly the same strength.

ent upon the strength of the batteries; the velocities from signals east-west and signals west-east coming out more and more nearly equal as the strength of the batteries was increased. See Dr. Gould's Report on telegraphic determinations of differences of longitude, in the Report of the Superintendent of the U. S. Coast Survey for 1857, Appendix No. 27.

FOURTH METHOD.—BY MOON CULMINATIONS.

229. The moon's motion in right ascension is so rapid that the change in this element while the moon is passing from one meridian to another may be used to determine the difference of longitude. Its right ascension at the instant of its meridian transit is most accurately found by means of the interval of sidereal time between this transit and that of a neighboring well-known star. For this purpose, therefore, the Ephemerides contain a list of *moon-culminating stars*, which are selected for each day so that at least four of them are given, the mean of whose declinations is nearly the same as that of the moon on that day, and, generally, so that two precede and two follow the moon. The Ephemerides also contain the right ascension of the moon's bright limb for each culmination, both upper and lower, and the variation of this right ascension in one hour of longitude, —*i.e.* the variation during the interval between the moon's transits over two meridians whose difference of longitude is one hour. This variation is not uniform, and its value is given for the instant of the passage over the meridian of the Ephemeris. These quantities facilitate the reduction of corresponding observations, as will be seen below.

230. As to the observation, let

$\vartheta, \vartheta' =$ the sidereal times of the culmination of the moon's limb and the star, respectively, corrected for all the known errors of the transit instrument, and for clock rate,

$a, a' =$ the right ascensions of the moon's limb and the star at the instants of transit;

then we evidently have

$$a = a' + \vartheta - \vartheta' \qquad (406)$$

232. *Corresponding observations at places whose difference of longitude is less than two hours.*—At each place the true sidereal times of transit of the moon-culminating stars and of the moon's bright limb are to be obtained with all possible precision: from these, according to the formula (406), will follow the right ascension of the moon's limb at the instants of transit over the two meridians, taking in each case the mean value found from all the stars observed.   Put

$L_1, L_2 = $ the approximate or assumed longitudes,
$\lambda = $ the true difference of longitude,
$a_1, a_2 = $ the observed right ascensions of the moon's bright limb at L_1 and L_2 respectively,
$H_0 = $ the variation of the R. A. of the moon's limb for 1^h of longitude while passing from L_1 to L_2;

then we have

$$\lambda = \frac{a_2 - a_1}{H_0} \qquad\qquad (407)$$

in which, $a_2 - a_1$ and H_0 being both expressed in seconds, λ will be in hours and decimal parts.

When the difference of longitude is less than two hours, it is found to be sufficiently accurate to regard H_0 as constant, provided we employ its value for the middle longitude $L_0 = \frac{1}{2}(L_1 + L_2)$, found by interpolation from the values in the Ephemeris, having regard to second differences.

EXAMPLE.—The following observations were made, May 15, 1851, at Santiago, Chili, by the U. S. Astronomical Expedition under Lieut. GILLISS, and at Philadelphia, by Prof. KENDALL:

Object.	Santiago sid. time.	Philad'a sid. time.
ϑ *Librae*	$15^h 46^m \ 3^s.37$	$15^h 45^m 22^s.33$
Moon II Limb	16 21 36 .84	16 21 39 .11
B. A. C. 5579	16 33 40 .12	16 32 58 .96

We shall assume the longitudes from Greenwich to be,

Philadelphia, $L_1 = 5^h \ 0^m \ 39^s.85$
Santiago, $L_2 = 4 \ 42 \ 19$.

the longitude of Philadelphia being that which results from the last chronometric expeditions of the U. S. Coast Survey, and that of Santiago the value which Lieut. GILLISS at first assumed.

in which the value of H must be that which belongs to the uncertain meridian L_2, or, more strictly, H must be taken for the mean longitude between L_2 and $L_2 + \Delta L$; but, as ΔL is generally very small, great precision in H is here superfluous. However, if in any case ΔL is large, we can first find H for the meridian L_2, and with this value an approximate value of ΔL; then, interpolating H for the meridian $L_2 + \frac{1}{2} \Delta L$, a more correct value of ΔL will be found.*

EXAMPLE.—The following observations were made May 15, 1851, at Santiago and Greenwich:

Object.	Santiago.	Greenwich.
ϑ Librae	15^h 46^m $3^s.37$	15^h 45^m $22^s.37$
Moon II Limb	16 21 36 .84	16 9 39 .41
B.A.C. 5579	16 33 40 .12	16 32 59 .17

We assume here, as in the preceding example, for Santiago $L_2 = 4^h 42^m 19^s$, and for Greenwich we have $L_1 = 0$. The places of the stars being as in the preceding article, we find for

$$\text{Greenwich, } a_1 = 16^h \ 9^m \ 39^s.54$$
$$\text{Santiago, } \quad a_2 = 16 \ 20 \ 55 .99$$
$$a_2 - a_1 = \quad \ \ 11 \ 16 .45$$

The computed right ascension for Greenwich is in this case simply that given in the Ephemeris for May 15; the increase to the meridian $4^h 42^m 19^s.0$ has been found in our example of interpolation, Art. 71, to be

$$A_2 - A_1 = 11^m \ 15^s.84$$

and hence

$$\gamma = + 0^s.61$$

We find, moreover, for the longitude $4^h 42^m 19^s$,

$$H = 143^s.77$$

whence

$$\Delta L = + 0^s.61 \times \frac{3600}{143.77} = + 15^s.28$$

By these observations we have, therefore,

$$\text{Longitude of Santiago} = 4^h \ 42^m \ 34^s.28$$

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* This method of reducing moon culminations was developed by WALKER, *Transactions of the American Philosophical Society*, new series, Vol. V.

or as follows: Let T_0 and $T_0 + 1^h$ be the two Greenwich hours between which α_1 falls, and put

$\Delta\alpha$ = the increase of right ascension in 1^s of mean time at the time T_0,

$\delta\alpha$ = the increase of $\Delta\alpha$ in 1^h,

α_0 = the right ascension of the Ephemeris at the hour T_0,

then, by the method of interpolation by second differences, we have

$$\alpha_1 = \alpha_0 + \left[\Delta\alpha + \frac{\delta\alpha}{2}\cdot\frac{T_1 - T_0}{3600}\right]\left(\frac{T_1 - T_0}{60}\right)$$

in which the interval $T_1 - T_0$ is supposed to be expressed in seconds. This gives

$$T_1 - T_0 = \frac{60\,(\alpha_1 - \alpha_0)}{\Delta\alpha + \dfrac{\delta\alpha}{2}\cdot\dfrac{T_1 - T_0}{3600}}$$

and in the second member an approximate value of T_1 may be used, deduced from the local time of the observation and an approximate longitude. A still more convenient form, which dispenses with finding an approximate value of T_1, is obtained as follows: Put

$$T_1 = T_0 + x$$

then we have

$$\frac{S}{15\,(1 - \lambda)\cos\delta}$$

in which S = the moon's semidiameter. λ = the increase of the moon's right ascension in one sidereal second, and δ = the moon's declination, which are to be taken for the Greenwich time of the observation, approximately known from the local time and the approximate longitude.

Or we may apply to the sidereal time ($= \vartheta_1$) of the transit of the limb the quantity

$$\frac{S}{15\cos\delta}$$

and the resulting $\alpha_1 = \vartheta_1 \pm \tfrac{1}{15} S \sec\delta$ will be the right ascension of the moon's centre at the local sidereal time ϑ_1. We then find the Greenwich time Θ_1 corresponding to α_1 as in the text, and we have

$$L_1 = \Theta_1 - \vartheta_1$$

$$x = \frac{60\,(a_1 - a_0)}{\Delta a \left(1 + \dfrac{x}{7200} \cdot \dfrac{\delta a}{\Delta a} \right)}$$

$$= \frac{60\,(a_1 - a_0)}{\Delta a} \left(1 + \frac{x}{7200} \cdot \frac{\delta a}{\Delta a} \right)^{-1}$$

or, with sufficient accuracy,

$$x = \frac{60\,(a_1 - a_0)}{\Delta a} \left(1 - \frac{x}{7200} \cdot \frac{\delta a}{\Delta a} \right)$$

Putting then

$$x' = \frac{60\,(a_1 - a_0)}{\Delta a} \qquad x'' = - \frac{x'^2}{7200} \cdot \frac{\delta a}{\Delta a} \qquad\qquad (412)$$

we have, very nearly,

$$x = x' + x'' \qquad\qquad (413)$$

As a practical rule for the computer, we may observe that x'' will be a positive quantity when Δa is decreasing, and negative when Δa is increasing.

The method of this article will be found particularly convenient when the observation is compared directly with the Ephemeris, the latter being corrected by the following process. See page 362.

235. *Peirce's method of correcting the Ephemeris.**—The accuracy of the longitude found by a moon culmination depends upon that of the observed difference of right ascension. When this difference is obtained from two corresponding observations, if the probable errors of the observed right ascensions at the two meridians are ϵ_1 and ϵ_2, the probable error of the difference will be $= \sqrt{(\epsilon_1^2 + \epsilon_2^2)}$. [Appendix]. But if instead of an actual observation at L_2 we had a perfect Ephemeris, or $\epsilon_2 = 0$, the probable error of the observed difference would be reduced to ϵ; and if we have an Ephemeris the probable error of which is less than that of an observation, the error of the observed difference is reduced. At the same time, we shall gain the additional advantage that every observation taken at the meridian whose longitude is required will become available, even when no corresponding observation has been taken on the same day; and

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* Report of the Superintendent of the U. S. Coast Survey for 1854, Appendix, p. 115*.

experience has shown that, when we depend on corresponding observations alone, about one-third of the observations are lost.

The defects of the lunar theory, according to PEIRCE, are involved in several terms which for each lunation may be principally combined into two, of which one is constant and the other has a period of about half a lunation, and he finds that for all practical purposes we may put the correction of the Ephemeris for each semi-lunation under the form

$$X = A + Bt + Ct^2 \qquad (414)$$

in which A, B, and C are constants to be determined from the observations made at the principal observatories during the semi-lunation, and t denotes the time reckoned from any assumed epoch, which it will be convenient to take near the mean of the observations. The value of t is expressed in days; and small fractions of a day may be neglected. Let

a_1, a_2, a_3, &c. = the right ascension observed at any observatory at the dates t_1, t_2, t_3, &c., from the assumed epoch,

a_1', a_2', a_3', &c. = the right ascension at the same instant found from the Ephemeris,

and put

$$n_1 = a_1 - a_1', \quad n_2 = a_2 - a_2', \quad n_3 = a_3 - a_3', \text{ &c.}$$

then n_1, n_2, n_3, &c. are the corrections which (according to the observations) the Ephemeris requires on the given dates, and hence we have the equations of condition

$$A + Bt_1 + Ct_1^2 - n_1 = 0$$
$$A + Bt_2 + Ct_2^2 - n_2 = 0$$
$$A + Bt_3 + Ct_3^2 - n_3 = 0$$
$$\text{&c.}$$

In order to eliminate constant errors peculiar to any observatory, when the observation is not made at Greenwich, the observed right ascension is to be increased by the average excess for the year (determined by simultaneous observations) of the right ascensions of the moon's limb made at Greenwich above those made at the actual place of observation.

If now we put

$m =$ the number of observations $=$ the number of equations of condition,
$T =$ the algebraic sum of the values of t,
$T_2 =$ the sum of the squares of t,
$T_3 =$ the algebraic sum of the third powers of t,
$T_4 =$ the sum of the fourth powers of t,
$N =$ the algebraic sum of the values of n,
$N_1 =$ the algebraic sum of the products of n multiplied by t,
$N_2 =$ the algebraic sum of the products of n multiplied by t^2,

the normal equations, according to the method of least squares, will be

$$\left. \begin{array}{l} mA + TB + T_2 C - N = 0 \\ TA + T_2 B + T_3 C - N_1 = 0 \\ T_2 A + T_3 B + T_4 C - N_2 = 0 \end{array} \right\} \quad (415)$$

The solution of these equations by the method of successive substitution, according to the forms given in the Appendix, may be expressed as follows:

$$T_2' = T_2 - \frac{T^2}{m} \qquad\qquad N_1' = N_1 - \frac{TN}{m}$$

$$T_3' = T_3 - \frac{TT_2}{m} \qquad\qquad N_2' = N_2 - \frac{T_2 N}{m}$$

$$T_4' = T_4 - \frac{T_2^2}{m} \qquad\qquad N_2'' = N_2' - \frac{T_3' N_1'}{T_2'}$$

$$T_4'' = T_4' - \frac{(T_3')^2}{T_2'}$$

$$C = \frac{N_2''}{T_4''} \qquad B = \frac{N_1' - T_3' C}{T_2'} \qquad A = \frac{N - T_2 C - TB}{m} \quad (416)$$

Then, to find the mean error of the corrected Ephemeris, we observe that this error is simply that of the function X, which is to be found by the method of the Appendix, according to which we first find the coefficients k_0, k_1, k_2 by the following formulæ:

$$mk_0 = 1$$
$$mk_0 + T_2' k_1 = t$$
$$mk_0 + T_3' k_1 + T_4'' k_2 = t^2$$

and then, putting

$$M = \sqrt{(k_0^2 m + k_1^2 T_2' + k_2^2 T_4'')}$$

we have

$$(\epsilon X) = M\epsilon \qquad (417)$$

in which ϵ denotes the mean error of a single observation and (ϵX) the mean error of the corrected Ephemeris; or, if ϵ denotes the probable error of an observation, (ϵX) denotes the probable error of the corrected Ephemeris. (Appendix.)

If the values of k_0, k_1, and k_2 are substituted in M, we shall have

$$M = \sqrt{\left[\frac{1}{m} + \frac{(t-1)^2}{T_2'} + \frac{(t-1)^2}{T_4''}\left(t + 1 - \frac{T_3'}{T_2'}\right)^2\right]} \qquad (418)$$

It will generally happen, where a sufficient number of observations are combined, that $\dfrac{T_3'}{T_2'}$ is a small fraction which may be neglected without sensibly affecting the estimation of a probable error, and we may then take

$$M = \sqrt{\left[\frac{1}{m} + \frac{(t-1)^2}{T_2'} + \frac{(t^2-1)^2}{T_4''}\right]} \qquad (418^*)$$

According to PEIRCE, the probable error of a *standard* observation of the moon's transit is $0^s.104$ (found from the discussion of a large number of Greenwich, Cambridge, Edinburgh, and Washington observations); so that the probable error of the corrected Ephemeris will be equal to $M.(0^s.104)$.

EXAMPLE.—At the Washington Observatory, the following right ascensions of the moon were obtained from the transits over twenty-five threads, observed with the electro-chronograph:

Approx. Green. Mean Time.	R. A. of ☽ II Limb.	Sid. time semid. passing merid.	R. A. of ☽ centre = a_1.
1859, Aug. 16, 19ʰ	0ʰ 8ᵐ 53ˢ.40	62ˢ.06	0ʰ 7ᵐ 51ˢ.34
" 17, 20	0 54 33 .57	63 .54	0 53 30 .03
" 18, 21	1 42 48 .53	65 .77	1 41 42 .76

The sidereal time of the semidiameter passing the meridian is here taken from the British Almanac, as we propose to reduce the observations by means of the Greenwich observations which are reduced by this almanac. We thus avoid any error in the semidiameter.

During the semi-lunation from Aug. 13 to Aug. 27, the Greenwich observations, also made with the electro-chronograph,

gave the following corrections ($= n$) of the Nautical Almanac right ascensions of the moon:

Approx. Greenwich Mean Time.	n	t
1859. Aug. 14, 13^h	— 0^s.39	— 3.
" 15, 14	— 0 .26	— 1.9
" 16, 14	— 0 .49	— 0.9
" 18, 16	— 0 .63	+ 1.2
" 19, 17	— 1 .04	+ 2.2
" 20, 17	— 1 .08	+ 3.2

Let us employ these observations to determine by Peirce's method the most probable correction of the Ephemeris on the dates of the Washington observations. Adopting as the epoch Aug. 17th 12^h or 17^d.5, the values of t are approximately as above given. The correction of the Ephemeris being sensibly constant for at least one hour, these values are sufficiently exact. We find then

$$T = 0.8 \quad \begin{vmatrix} T_2 = & 29.94 \\ T_2' = & 29.83 \\ N_1 = & -3^s.89 \end{vmatrix} \begin{vmatrix} T_3 = & 10.556 \\ T_3' = & 6.564 \\ N_1 = & -4^s.41 \\ N_1' = & -3 .89 \end{vmatrix} \begin{vmatrix} T_4 = & 225.045 \\ T_4' = & 75.644 \\ N_2 = & -21^s.85 \\ N_2' = & - 2 .44 \end{vmatrix} T_4'' = 74.200$$

$$m = 6 \qquad N = -3^s.89 \qquad N_2'' = -1^s.58$$

and hence, by (416),

$$C = -0^s.02135 \qquad B = -0^s.1257 \qquad A = -0^s.525$$

The correction of the Ephemeris for any given date t, reckoning from Aug. 17.5, is, therefore,

$$X = -0^s.525 - 0^s.1257\, t - 0^s.02135\, t^2$$

Consequently, for the dates of the Washington observations, the correction and the probable error ($M\varepsilon$) of the correction, found by (418) or (418*), are as follows:

Aug. 16, 19^h	$t = -0.7$	$X = -0^s.45$	$M\varepsilon = 0^s.05$
17, 20	$t = +0.3$	$X = -0 .56$	$M\varepsilon = 0 .04$
18, 21	$t = +1.4$	$X = -0 .74$	$M\varepsilon = 0 .04$

The longitude of the Washington Observatory may now be found by the hourly Ephemeris (after applying these corrections), by the method of Art. 234. Taking the observation of Aug. 16, we have

Aug. 16, $T_0 = 19^h$, R. A. of Ephemeris $= 0^h 6^m 47^s.56$

$$X = \quad -0.45$$

$\Delta a = 1.8122 \quad \delta a = + 0.0023 \quad a_0 = 0 \ 6 \ 47.11$

$$a_1 = 0 \ 7 \ 51.34$$

$$a_1 - a_0 = \quad 1 \quad 4.23$$

$\log (a_1 - a_0)$ 1.80774	$\log x'^2$ 6.6554
ar. co. $\log \Delta a$ 9.74179	$\log \delta a$ 7.3617
$\log 60$ 1.77815	ar. co. $\log \Delta a$ 9.7418
$\log x'$ 3.32768	$\log \frac{1}{7200}$ 6.1427
$x' = 35^m 26^s.57$	$\log x''$ $n9.9016$
$x'' = - \quad 0.80$	
$x = 35 \ 25.77$	

Hence, Greenwich mean time $= T_0 + x = 19^h 35^m 25^s.77$

Sidereal time mean noon	$=$	9 37 24.18
Correction for $19^h 35^m 25^s.77$	$=$	3 13.09
Greenwich sidereal time	$=$	5 16 3.04
Local sidereal time $= a_1$	$=$	0 7 51.34
Longitude $=$		5 8 11.70

The observations of the 17th and 18th being reduced in the same manner, the three results are

		Probable error.*	Weight.
Aug. 16,	$5^h 8^m 11^s.70$	$3^s.5$	1.
" 17,	12.50	3.1	1.3
" 18,	11.10	2.9	1.5
Mean by weights $=$	5 8 11.74	1.8	

236. *Combination of moon culminations by weights.*—When some of the transits either of the moon or of the comparison stars are incomplete, one or more of the threads being lost, such observations should evidently have less weight than complete ones, if we wish to combine them strictly according to the theory of probabilities. Besides, other things being equal, a determination of the longitude will have more or less weight according to the greater or less rapidity of the moon's motion in right ascension.

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* For the computation of the probable error and weight, see the following article.

If the weight of a transit either of the moon or a star were simply proportional to the number of observed threads, as has been assumed by those who have heretofore treated of this subject,* the methods which they have given, and which are obvious applications of the method of least squares, would be quite sufficient. But the subject, strictly considered, is by no means so simple.

Let us first consider the formula

$$a_1 = a' + \vartheta_1 - \vartheta'$$

or, rather

$$a_1 = \vartheta_1 + (a' - \vartheta')$$

in which ϑ_1 and ϑ' are the observed sidereal times of the transit of the moon and star, respectively; a' is the tabular right ascension of the star, and a_1 is the deduced right ascension of the moon. The probable error of a_1 is composed of the probable errors of ϑ_1 and of $a' - \vartheta'$, which belong respectively to the moon and the star. We may here disregard the clock errors, as well as the unknown instrumental errors, since they affect ϑ_1 and ϑ' in the same manner, very nearly, and are sensibly eliminated in the difference $\vartheta_1 - \vartheta'$. The probable error of the quantity $a' - \vartheta'$ is composed of the errors of a' and ϑ'. The probable error of the tabular right ascension of the moon-culminating stars is not only very small, but in the case of corresponding observations is wholly eliminated; and even when we use a corrected Ephemeris it will have but little effect, since the observed right ascension of the moon at the principal observatories always depends (or at least should depend) chiefly upon these stars. We may, therefore, consider the error of $a' - \vartheta'$ as simply the error of ϑ'. We have here to deal with those errors only which do not necessarily affect ϑ' and ϑ_1 in the same manner, and of these the chief and only ones that need be considered here are—1st, the *culmination error* produced by the peculiar conditions of the atmosphere at the time of the star's transit, which are constant, or nearly so, during the transit, but are different for different stars and on different days; and, 2d, the *accidental error of observation*. It is only the latter which can be diminished

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* NICOLAI, in the *Astronomische Nachrichten*, No. 26; and S. C. WALKER, Transactions of the American Philosophical Society, Vol. VI. p. 258.

by increasing the number of threads. In Vol. II. (Transit Instrument) I shall show that the probable error of a single determination of the right ascension of an equatorial star (and this may embrace the moon-culminating stars) at the Greenwich Observatory is $0^{s}.06$, whereas, if the culmination error did not exist it would be only $0^{s}.03$, the probable error of a single thread being $= 0^{s}.08$, and the number of threads $= 7$. Hence, putting

$c =$ the probable culmination error for a star,

we deduce*

$$c = \sqrt{(0.06)^2 - (0.03)^2} = 0^{s}.052$$

If, then, we put

$\epsilon =$ the probable accidental error of the transit of a star over a single thread,

$n =$ the number of threads on which the star is observed,

the probable error of ϑ', and, consequently, also of $\alpha' - \vartheta'$, is

$$= \sqrt{c^2 + \frac{\epsilon^2}{n}}$$

and the weight of $\alpha' - \vartheta'$ for each star may be found by the formula

$$p = \frac{E^2}{c^2 + \frac{\epsilon^2}{n}}$$

in which E is the probable error of an observation of the weight unity, which is, of course, arbitrary. If we make $p = 1$ when $n = 7$, we have $E = 0^{s}.06$. Substituting this value, and also $c = 0^{s}.052$, $\epsilon = 0^{s}.08$, the formula may be reduced to the following:

$$p = \frac{134}{100 + \frac{238}{n}} \tag{419}$$

The value of α_1 is to be deduced by adding to ϑ_1 the mean

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* The value of c thus found involves other errors besides the culmination error proper, such as unknown irregularities of the clock and transit instrument, &c. These cannot readily be separated from c, nor is it necessary for our present purpose.

and the probable error of the deduced longitude will be

$$= h \sqrt{\epsilon_1^2 + \epsilon_2^2} \qquad (423)$$

where, H being the increase of the moon's right ascension in 1^h of longitude, we have

$$h = \frac{3600}{H} \qquad (424)$$

But if the observation at the meridian L_1 is compared with a corrected Ephemeris (Art. 235) the probable error of which is $M(0^s.104)$, the probable error of the deduced longitude will be

$$= h \sqrt{\epsilon_1^2 + M^2 (0.104)^2} \qquad (425)$$

Finally, all the different values of the longitude will be combined by giving them weights reciprocally proportional to the squares of their probable errors.

The preponderating influence of the constant error represented by the first term of (422) is such that a very precise evaluation of the other terms is quite unimportant. It is also evident that we shall add very little to the accuracy of an observation by increasing the number of threads of the reticule beyond five or seven. For example, suppose, as in the Washington observations used in Art. 235, that twenty-five threads are taken, and that four stars are compared with the moon; we have for each star, by (419),

$$p = \frac{134}{100 + \frac{238}{25}} = 1.22$$

and hence

$$\epsilon_1 = \sqrt{\left[(0.091)^2 + \frac{(0.104)^2}{25} + \frac{(0.06)^2}{4.88}\right]} = 0^s.097$$

whereas for seven threads we have $\epsilon_1 = 0^s.104$, and therefore the increase of the number of threads has not diminished the probable error by so much as $0^s.01$.

For the observations of 1859 August 16, 17, 18, Art. 235, the values of h are respectively

$$32.1 \qquad 30.8 \qquad \text{and} \qquad 28.8$$

and, taking $M\epsilon = M(0^s.104)$ as given in that article, namely,

$$0^s.05 \qquad 0^s.04 \qquad \text{and} \qquad 0^s.04$$

with the value of $\varepsilon_1 = 0'.097$ above found, we deduce the probable errors of the three values of the longitude, by (425),

$$3'.5 \qquad 3'.1 \qquad \text{and} \qquad 2'.9$$

The reciprocals of the squares of these errors are very nearly in the proportion of the numbers 1, 1.3, 1.5, which were used as the weights in combining the three values.

237. The advantage of employing a corrected Ephemeris instead of corresponding observations can now be determined by the above equations. If the observations are all *standard* observations (represented by $n_1 = 7$ and $[p] = 4$), we shall have $\varepsilon_1 = \varepsilon_2 = 0'.104$, and the probable error of the longitude will be

$$\text{by corresponding observations} = h\varepsilon_1 \sqrt{2}$$
$$\text{by the corrected Ephemeris} \quad = h\varepsilon_1 \sqrt{1 + M^2}$$

The latter will, therefore, be preferable when $M < 1$, which will always be the case except when very few observations have been taken at the principal observatories.

But experience has shown that when we depend wholly on corresponding observations we lose about one-third of the observations, and, consequently, the probable error of the final longitude from a series of observations is greater than it would be were all available in the ratio of $\sqrt{3} : \sqrt{2}$. Hence the probable errors of the final results obtained by corresponding observations exclusively, and by employing the corrected Ephemeris by which all the observations are rendered available, are in the ratio $\sqrt{3} : \sqrt{1 + M^2}$, and, the average value of M being about 0.6, this is as $1 : 0.67$.

If, however, on the date of any given observation at the meridian to be determined, we can find corresponding observations at *two* principal observatories, the probable error of the longitude found by comparing their mean with the given observation will be only $h\varepsilon_1 \sqrt{1.5}$, which is so little greater than the average error in the use of the corrected Ephemeris, that it will hardly be worth while to incur the labor attending the latter. If there should be *three* corresponding observations, the error will be reduced to $h\varepsilon_1 \sqrt{1.33}$, and, therefore, less than the average error of the corrected Ephemeris.

vivifying influence will prevent science from stagnating into mere mechanical drudgery.

"In approaching the ultimate limit of accuracy, the probable error ceases to diminish proportionably to the increase of the number of observations, so that the accuracy of the mean of several determinations does not surpass that of the single determinations as much as it should do in conformity with the law of least squares: thus it appears that the probable error of the mean of the determinations of the longitude of the Harvard Observatory, deduced from the moon-culminating observations of 1845, 1846, and 1847, is 1'.28 instead of being 1'.00, to which it should have been reduced conformably to the accuracy of the separate determinations of those years.

"One of the fundamental principles of the doctrine of probabilities is, that the probability of an hypothesis is proportionate to its agreement with observation. But any supposed computed lunar epoch may be changed by several hundredths of a second without perceptibly affecting the comparison with observation, provided the comparison is restricted within its legitimate limits of tenths of a second. Observation, therefore, gives no information which is opposed to such a change."

The ultimate limit of accuracy in the determination of a longitude by moon culminations, according to the same distinguished authority, is *not less than one second of time.* This limit can probably be reached by the observations of two or three years, if all the possible ones are taken; and a longer continuance of them would be a waste of time and labor.

From these considerations it follows that the method of moon culminations, when the transits of the limb are employed, cannot come into competition with the methods by chronometers and occultations where the latter are practicable.*

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* In consequence of the uncertainty attending the observation of the transit of the moon's limb, it has been proposed by MAEDLER (*Astron. Nach.* No. 837) to substitute the transit of a well-defined lunar spot. The only attempt to carry out this suggestion, I think, is that of the U. S. Coast Survey, a report upon which by Mr. PETERS will be found in the Report of the Superintendent for 1856, p. 198. The varying character of a spot as seen in telescopes of different powers presents, it seems to me, a very formidable obstacle to the successful application of this method.

FIFTH METHOD.—BY AZIMUTHS OF THE MOON, OR TRANSITS OF THE
MOON AND A STAR OVER THE SAME VERTICAL CIRCLE.

239. The travelling observer, pressed for time, will not unfre-
quently find it expedient to mount his transit instrument in the
vertical circle of a circumpolar star, without waiting for the meri-
dian passage of such a star. The methods of determining the
local time and the instrumental constants in this case are given
in Vol. II. He may then also observe the transit of the moon
and a neighboring star, and hence deduce the right ascension of
the moon, which may be used for determining his longitude
precisely as the culminations are used in Art. 234.

240. But if the local time is previously determined, we may
dispense with all observations except those of the moon and the
neighboring star, and then we can repeat the observation several
times on the same night by setting the instrument successively
in different azimuths on each side of the meridian. It will not
be advisable to extend the observations to azimuths of more than
15° on either side.

The altitude and azimuth instrument is peculiarly adapted for
such observations, as its horizontal circle enables us to set it at
any assumed azimuth when the direction of the meridian is
approximately known. The zenith telescope will also answer
the same purpose. But as the horizontal circle reading is not
required further than for setting the instrument, it is not indis-
pensable, and therefore the ordinary portable transit instrument
may be employed, though it will not be so easy to identify the
comparison star.

The comparison star should be one of the well-determined
moon-culminating stars, as nearly as possible in the same
parallel with the moon, and not far distant in right ascension,
either preceding or following.

The chronometer correction and rate must be determined, with
all possible precision, by observations either before or after the
moon observations, or both. An approximate value of the cor-
rection should be known before commencing the observations,
as it will be expedient to compute the hour angles and zenith
distances of the two objects for the several azimuths at which it
is proposed to observe, in order to point the instrument properly
and thus avoid observing the wrong star.

We shall make use of the following notation:

$T, T' =$ the mean of the chronometer times of transit of the moon's limb and the star, respectively, over the several threads,*

$\Delta T, \Delta T' =$ the corresponding chronometer corrections,

$b, b' =$ the inclinations of the horizontal axis at the times T and T',

$c =$ the collimation constant for the mean of the threads,

$a, a' =$ the moon's and the star's right ascensions,

$\delta, \delta' = $ "   "   "   declinations,

$t, t' =$ "   "   "   hour angles,

$\zeta, \zeta' =$ "   "   "   true zenith distances,

$q, q' =$ "   "   "   parallactic angles,

$A, A' =$ "   "   "   azimuths,

$\Delta a =$ the increase of the moon's right ascension in one minute of mean time,

$\Delta \delta =$ the increase (positive towards the north) of the moon's declination in one minute of mean time,

$\pi =$ the moon's equatorial horizontal parallax,

$S =$ the moon's geocentric semidiameter,

$\varphi =$ the observer's latitude,

$L' =$ the assumed longitude,

$\Delta L =$ the required correction of this longitude,

$L =$ the true longitude $= L' + \Delta L$.

The moon's a, δ, π, and S are to be taken from the Ephemeris for the Greenwich time $T + \Delta T + L'$ (expressed in mean time). The changes Δa, $\Delta \delta$ are also to be reduced to this time. The right ascension and declination must be accurately interpolated, from the hourly Ephemeris, with second differences.

The quantities A, ζ, q are now to be computed for the chronometer time T, and A', ζ', q' for the time T'. Since A and A'

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* The chronometer time of passage over the mean of the threads will be obtained rigorously by reducing each thread separately to the mean of all by the general formula given for the purpose in Vol. II. If, however, the same threads are employed for both moon and star, and c denotes the equatorial distance of the mean of the actually observed threads from the collimation axis, it will suffice (unless the observations are extended greatly beyond the limits recommended in the text) to take the means of the observed times at the times of passage over the fictitious thread the collimation of which is $= c$. The slight theoretical error which this procedure involves will be eliminated if the observations are arranged symmetrically with respect to the meridian.

are required with all possible precision, logarithms of at least six decimal places are to be employed in their computation; but for ζ, q, ζ', q', four decimal places will suffice. The following formulæ for this purpose result from a combination of (16) and (20):

<div align="center">For the moon.   For the star.</div>

$$t = T + \Delta T - a \qquad\qquad t' = T' + \Delta T' - a'$$

$$\left.\begin{aligned}\tan M &= \tan \delta \sec t \\[1mm] \tan A &= \frac{\tan t \cos M}{\sin (\varphi - M)}\end{aligned}\right\} \text{with six} \atop \text{decimals;} \left\{\begin{aligned}\tan M' &= \tan \delta' \sec t' \\[1mm] \tan A' &= \frac{\tan t' \cos M'}{\sin (\varphi - M')}\end{aligned}\right.$$

<div align="right">(426)</div>

$$\left.\begin{aligned}\tan N &= \cot \varphi \cos t \\[1mm] \tan q &= \frac{\tan t \sin N}{\cos (\delta + N)} \\[1mm] \tan \zeta &= \frac{\cot (\delta + N)}{\cos q}\end{aligned}\right\} \text{with four} \atop \text{decimals;} \left\{\begin{aligned}\tan N' &= \cot \varphi \cos t' \\[1mm] \tan q' &= \frac{\tan t' \sin N'}{\cos (\delta' + N')} \\[1mm] \tan \zeta' &= \frac{\cot (\delta' + N')}{\cos q'}\end{aligned}\right.$$

in which A and q are to be so taken that $\sin A$ and $\sin q$ shall have the same sign as $\sin t$.

The true azimuth of the moon's limb will be found by applying to the azimuth of the centre the correction

$$\pm \frac{S}{\sin \zeta} \left[\begin{array}{l}\text{upper sign for 1st limb} \\ \text{lower " " 2d "}\end{array}\right]$$

If we assume the parallax of the limb to be the same as that of the centre (which involves but an insensible error in this case), we next find the apparent azimuth of the limb by applying the correction given by (116), or

$$\rho \pi (\varphi - \varphi') \sin 1'' \sin A' \operatorname{cosec} \zeta$$

in which $\varphi - \varphi'$ is the reduction of the latitude, and ρ is the terrestrial radius for the latitude φ. In this expression we employ A, which is the computed azimuth of the star, for the apparent azimuth of the moon's limb, since by the nature of the observation they are very nearly equal.

To correct strictly for the collimation and level of the instrument, we must have the moon's and star's apparent zenith distances, which will be found with more than sufficient accuracy for the purpose by the formulæ

<div align="center">moon's app. zen. dist. $= \zeta_1 = \zeta + \pi \sin \zeta -$ refraction
star's " " " $= \zeta_1' = \zeta' -$ refraction</div>

and then the reduction of the true azimuth to the instrumental azimuth (see Vol. II., *Altitude and Azimuth Instrument*) is

$$\text{for the moon,} \quad \mp \frac{c}{\sin \zeta_1} \mp \frac{b}{\tan \zeta_1}$$

$$\text{for the star,} \quad \mp \frac{c}{\sin \zeta_1'} \mp \frac{b'}{\tan \zeta_1'}$$

the upper or lower sign being used according as the vertical circle is on the left or the right of the observer. The computed instrumental azimuths are, therefore,

$$\text{(moon)} \ A_1 = A \pm \frac{S}{\sin \zeta} + \frac{\rho\pi(\varphi-\varphi')\sin 1'' \sin A'}{\sin \zeta} \mp \frac{c}{\sin \zeta_1} \mp \frac{b}{\tan \zeta_1}$$

$$\text{(star)} \ A_1' = A' \mp \frac{c}{\sin \zeta_1'} \mp \frac{b'}{\tan \zeta_1'} \qquad (427)$$

If now the longitude and other elements of the computation are correct, we shall find A_1 and A_1' to be equal: otherwise, put

$$x = A_1 - A_1' \qquad (428)$$

then we are to find how the required correction $\triangle L$ depends on x, supposing here that all the elements which do not involve the longitude are correct. Now, we have taken α and δ from the Ephemeris for the Greenwich sidereal time $T + \triangle T + L'$, when they should be taken for the time $T + \triangle T + L' + \triangle L$. Hence, if λ and β denote the increments of the moon's right ascension and declination in one sidereal second, both expressed in seconds of arc,

$$\lambda = \frac{15 \triangle \alpha}{60.164} = [9.39675] \triangle \alpha$$

$$\beta = \frac{\triangle \delta}{60.164} = [8.22066] \triangle \delta \qquad (429)$$

we find that

$$\alpha \text{ requires the correction} \quad \lambda . \triangle L$$
$$\delta \quad \text{``} \quad \text{``} \quad \beta . \triangle L$$
$$t \quad \text{``} \quad \text{``} \quad -\lambda . \triangle L$$

and these corrections must produce the correction $-x$ in the moon's azimuth. The relations between the corrections of the azimuth, the hour angle, and the declination, where these are so small as to be treated as differentials, is, by (51),

$$dA = \frac{\cos \delta \cos q}{\sin \zeta} dt + \frac{\sin q}{\sin \zeta} d\delta$$

that is,

$$-x = -\frac{\cos \delta \cos q}{\sin \zeta} \lambda . \Delta L + \frac{\sin q}{\sin \zeta} \beta . \Delta L$$

Hence, if we put

$$a = \lambda . \frac{\cos \delta \cos q}{\sin \zeta} - \beta . \frac{\sin q}{\sin \zeta} \qquad (430)$$

we have

$$\Delta L = \frac{x}{a} \qquad (431)$$

and hence, finally, the true longitude $L' + \Delta L$.

241. In order to determine the relative advantages of this method and that of meridian transits, let us investigate a formula which shall exhibit the effect of every source of error. Let

$\delta a, \delta \delta, \delta \pi, \delta S =$ the corrections of the elements taken from the Ephemeris of the moon,

$\delta a', \delta \delta' =$ the corrections of the star's place,

$\delta T, \delta T' =$ the corrections for error in the obs'd time,

$\delta \Delta T =$ the correction of ΔT,

$\delta \varphi =$ the correction of φ.

If, when the corrected values of all the elements—that of the longitude included—are substituted in the above computation, A_1 and A_1' become $A_1 + dA_1$ and $A_1' + dA_1'$, we ought to find, rigorously,

$$A_1 + dA_1 = A_1' + dA_1'$$

which compared with (428) gives

$$x = - dA_1 + dA_1' \qquad (432)$$

We have, therefore, to find expressions for dA_1 and dA_1' in terms of the above corrections and of ΔL. We have, first, by differentiating (427),

$$dA_1 = dA \pm \frac{dS}{\sin \zeta} + \frac{\rho (\varphi - \varphi') \sin 1'' \sin A'}{\sin \zeta} d\pi$$

$$dA_1' = dA'$$

We neglect errors in c and b which are practically eliminated by comparing the moon with a star of nearly the same declination, and combining observations in the reverse positions of the axis.

The total differential of A is, by (51), after reducing dt to arc,

$$dA = \frac{\cos \delta \cos q}{\sin \zeta} \cdot 15 \, dt + \frac{\sin q}{\sin \zeta} \, d\delta - \cot \zeta \sin A \, d\varphi$$

consequently, also,

$$dA' = \frac{\cos \delta' \cos q'}{\sin \zeta'} \cdot 15 dt' + \frac{\sin q'}{\sin \zeta'} \, d\delta' - \cot \zeta' \sin A' d\varphi$$

Since $t = T + {}_\Delta T - a$, we have

$$dt = dT + d_\Delta T - da$$

where dT and $d_\Delta T$ may be at once exchanged for δT and $\delta_\Delta T$; but da is composed of two parts: 1st, the correction of the Ephemeris, and 2d, $\lambda ({}_\Delta L + \delta T + \delta_\Delta T)$, which results from our having taken a for the uncorrected time. Hence we have, in arc,

$$15 \, dt = 15 \, \delta T + 15 \, \delta_\Delta T - 15 \, \delta a - \lambda ({}_\Delta L + \delta T + \delta_\Delta T)$$

The correction $d\delta$ is likewise composed of two parts, namely,

$$d\delta = \delta\delta + \beta ({}_\Delta L + \delta T + \delta_\Delta T)$$

Further, we have simply $d\delta' = \delta\delta'$ and

$$dt' = \delta T' + \delta_\Delta T' - \delta a'$$

but, as we may neglect the error in the rate of the chronometer for the brief interval between the observation of the moon and the star, we can take $\delta_\Delta T'' = \delta_\Delta T$, and, consequently,

$$dt' = \delta T' + \delta_\Delta T - \delta a'$$

When the substitutions here indicated are made in (432), we obtain the expression

$$x = a_\Delta L + 15 f \cdot \delta a - \frac{\sin q}{\sin \zeta} \cdot \delta\delta - (15 f - a) \delta T$$

$$- 15 f' \cdot \delta a' + \frac{\sin q'}{\sin \zeta'} \cdot \delta\delta' + 15 f' \cdot \delta T'$$

$$\mp \frac{\delta S}{\sin \zeta} - \frac{\rho (\varphi - \varphi') \sin 1'' \sin A'}{\sin \zeta} \delta\pi$$

$$- [15 (f - f') - a] \delta_\Delta T + \frac{\sin (\zeta' - \zeta) \sin A'}{\sin \zeta \sin \zeta'} \delta\varphi \qquad (433)$$

·in which the following abbreviations are used:

$$f = \frac{\cos \delta \cos q}{\sin \zeta} \qquad\qquad f' = \frac{\cos \delta' \cos q'}{\sin \zeta'}$$

$$a = \lambda f - \beta \frac{\sin q}{\sin \zeta}$$

and in the coefficient of $\delta\varphi$ we have put $A = A'$.

By the aid of this equation we can now trace the effect of each source of error.

1st. The coefficients of $\delta\delta$, $\delta\delta'$, $\delta\pi$, $\delta\varphi$ have different signs for observations on different sides of the meridian, and therefore the errors of declination, parallax, and latitude will be eliminated by taking the mean of a pair of observations equidistant from the meridian.

2d. The star's declination being nearly equal to that of the moon, we shall have very nearly $f = f'$, and the coefficient of $\delta\Delta T$ will be $= a$; and since to find ΔL we have yet to divide the equation by a, it follows that an error in the assumed clock correction produces an equal error (but with a different sign) in the longitude, as in the case of meridian observations.

3d. An error δT in the observed time of the moon's transit produces in the longitude the error

$$\left(\frac{15 f}{a} - 1 \right) \delta T$$

The mean of the values of a for two observations equidistant from the meridian is λf. The mean effect of the error δT is therefore

$$\left(\frac{15}{\lambda} - 1 \right) \delta T$$

which is the same as in the case of a meridian observation.

The effect of an error $\delta T'$ in the observed time of the star's transit is

$$\frac{15 f'}{a} \delta T'$$

and for two observations equidistant from the meridian, the star being in the same parallel as the moon, the mean effect is

$$\frac{15}{\lambda} \delta T'$$

also the same as for a meridian observation.

4th. An error δS in the tabular semidiameter is always eliminated in the case of meridian observations when they are compared with observations at another meridian, since the same semidiameter is employed in reducing the observations at both meridians. But in the case of an extra-meridian observation the effect upon the longitude is

$$\frac{\delta S}{a \sin \zeta} = \frac{\delta S}{\lambda \cos \delta \cos q - \beta \sin q}$$

and in the mean of two observations equidistant from the meridian, the values of q being small, it is

$$\frac{\delta S}{\lambda \cos \delta \cos q} = \frac{\delta S}{\lambda \cos \delta} (1 + 2 \sin^2 \tfrac{1}{2} q) \text{ nearly.}$$

For a meridian observation the error will be

$$\frac{\delta S}{\lambda \cos \delta}$$

The error in the case of extra-meridian observations, therefore, remains somewhat greater than in the case of meridian ones, the excess being nearly

$$\frac{2 \delta S . \sin^2 \tfrac{1}{2} q}{\lambda \cos \delta}$$

which, however, is practically insignificant; for we have not to fear that δS can be as great as $1''$, and therefore, taking $q = 15°$, $\delta = 30°$, and $\lambda = 0.4$, which are extreme values, the difference cannot amount to $0'.1$ in the longitude.

5th. The error δa of the tabular right ascension of the moon produces in the longitude the error

$$- \frac{15 f}{a} \delta a$$

and from the mean of two observations equidistant from the meridian, the error is

$$- \frac{15 \delta a}{\lambda}$$

as in the case of the meridian observation.

The error $\delta a'$ in the star's right ascension produces the error $\frac{15 \delta a'}{\lambda}$ when the star is in the same parallel as the moon.

From this discussion it follows that, by arranging the observations *symmetrically* with respect to the meridian, the mean result will be liable to no sensible errors which do not equally affect meridian observations. But for the large culmination error in the case of the moon (Art. 236), which equally affects extra-meridian observations, the latter would have a great advantage by diminishing the effect of accidental errors. But the probable error of the mean of two observations equidistant from the meridian, seven threads being employed, will be, by (422),

$$\epsilon_1 = \sqrt{\left[(0.091)^2 + \frac{(0.104)^2}{14} + \frac{(0.06)^2}{2}\right]} = 0^s.10$$

and that of a single meridian observation, *even where only one star is compared with the moon*, is, by the same formula, $= 0^s.11$. When we take into account the extreme simplicity of the computation, the method of moon culminations must evidently be preferred; and that of extra-meridian observations will be resorted to only in the case already referred to (Art. 239), where the traveller may wish to determine his position in the shortest possible time and without waiting to adjust his instrument accurately in the meridian.

EXAMPLE.—At the U. S. Naval Academy, 1857 May 9, I observed the following transits of the moon's second limb and of *σ Scorpii*, at an approximate azimuth of 10° East, with an ERTEL universal instrument of 15 inches focal length:

	Chronometer.	Level.	Collim.	
☽ II Limb.	$T = 16^h 11^m 30^s.17$	$b = + 2''.2$	$c = 0.0$	} Vertical circle
σ Scorpii	$T' = 16\ 27\ 49.83$	$b' = + 2\ .2$		} left.

These times are the means of three threads. The chronometer correction, found by transits of stars in the meridian, was $- 55^m 9^s.16$ at 13^h sidereal time, and its hourly rate $- 0^s.32$. The assumed latitude and longitude were

$$\varphi = 38° 58' 53''.5 \qquad\qquad L' = 5^h 5^m 55^s$$

The star's place was

$$\alpha' = 16^h 12^m 31^s.90 \qquad\qquad \delta' = - 25° 14' 58''.5$$

We first find the sidereal times of the observations of the moon and star respectively, and the Greenwich mean time of the observation of the moon: we have

$$\Delta T = - \quad 55^m \quad 9^s.89 \qquad\qquad \Delta T' = - \quad 55^m \quad 9^s.97$$
$$T + \Delta T = 15^h \ 16^m \ 20^s.28 \qquad T' + \Delta T' = 15^h \ 32^m \ 39^s.86$$
$$L' = \ 5 \quad 5 \quad 55 \ .$$

Gr. sidereal time =	20	22	15 .28
Sid. time Gr. moon =	3	8	58 .91
Sidereal interval =	17	13	16 .37
Red. to mean time = −		2	49 .28
Gr. mean time = May 9,	17^h	10^m	$27^s.09$

Hence from the Ephemeris we find

$$\alpha = 15^h \ 54^m \ 45^s.32 \qquad\qquad \delta = - \ 24° \ 42' \ 54''.4$$
$$\Delta\alpha = 2^s.1135 \qquad\qquad\qquad \Delta\delta = - \ 7''.619$$
$$S = 14' \ 47''.2 \qquad\qquad\qquad \pi = \quad 54' \ 9''.2$$

By (426) we find

$A = - \ 9° \ 40' \ 51''.2$	$A' = - \quad 9° \ 57' \ 14''.8$	
$\log \sin q = \quad n9.1581$	$\log \sin q' = \quad n9.1719$	
$\zeta = \quad 64° \ 19'.5$	$\zeta' = \quad 64° \ 54'.1$	
$\pi \sin \zeta = \quad + \ 48.8$		
Refraction $= \quad - \quad 2.1$	Refraction $= \quad - \quad 2.1$	
$\zeta_1 = \quad 65 \quad 6.2$	$\zeta_1' = \quad 64 \quad 52.0$	

For the latitude φ we find, from Table III.,

$$\log \rho = 9.9994 \qquad\qquad \varphi - \varphi' = 11' \ 15''$$

and then, by (427), we find

$$A = -9° \ 40' \ 51''.2 \qquad\quad A' = - 9° \ 57' \ 14''.8$$

$$-\frac{S}{\sin \zeta} = - \quad 16 \ 24 .2$$

$$\frac{\rho \pi \, (\varphi - \varphi') \sin 1'' \sin A'}{\sin \zeta} = - \quad\quad 2 .0$$

$$\frac{c}{\sin \zeta_1} = \quad\quad 0 .0 \qquad -\frac{c}{\sin \zeta_1'} = \quad\quad 0 .0$$

$$\frac{b}{\tan \zeta_1} = - \quad\quad 1 .0 \qquad -\frac{b'}{\tan \zeta_1'} = - \quad\quad 1 .0$$

$$A_1 = - \ 9 \ 57 \ 18 .4 \qquad\quad A_1' = - \ 9 \ 57 \ 15 .8$$

whence
$$x = -2''.6$$

By (429), (430), and (431), we find

$$\log \lambda = 9.72175 \qquad \log \beta = n9.10256 \qquad a = 0.5054$$

$$\Delta L = \frac{-2.6}{0.5054} = -5^{s}.15$$

If we wish to see the effect of all the sources of error in this example, we find, by (433),

$$0.5054 \, \Delta L = -2''.6 - 14.96 \, \delta\alpha - 0.16 \, \delta\delta + 14.45 \, \delta T - 14.82 \, \delta T' - 0.36 \, \delta\Delta T$$
$$+ 14.82 \, \delta\alpha' + 0.16 \, \delta\delta' + 1.11 \, \delta S - 0.001 \, \delta\pi + 0.004 \, \delta\varphi$$

The proper combination of observations is supposed to eliminate, or at least reduce to a minimum, all the errors except that of the moon's right ascension as given in the Ephemeris. In practice, therefore, it will be necessary to retain the term involving $\delta\alpha$. Thus, in the present case we take only

$$0.5054 \, \Delta L = -2''.6 - 14.96 \, \delta\alpha$$

A second observation on the same day at an azimuth 10° west gave

$$0.5458 \, \Delta L = -5''.7 - 14.92 \, \delta\alpha$$

The elimination of the errors of declination requires that we take the arithmetical mean of these equations; whence we have, finally,

$$\Delta L = -7^{s}.89 - 28.43 \, \delta\alpha$$

SIXTH METHOD.—BY ALTITUDES OF THE MOON.

242. The hour angle (t) of the moon may be computed from an observed altitude, the latitude and declination being known, and hence with the local sidereal time of the observation ($= \Theta$) the moon's right ascension by the equation $\alpha = \Theta - t$, with which the Greenwich time can be found, as in Art. 234, and, consequently, also the longitude.

The hour angle is most accurately found from an altitude when the observed body is on the prime vertical, and more accurately in low latitudes than in high ones (Art. 149). This method, therefore, is especially suited to low latitudes.

The method may be considered under two forms:—(A) that in which the moon's absolute altitude is directly observed and

employed in the computation of the hour angle; and (B) that in which the moon's altitude is compared differentially with that of a neighboring star,—*i.e.* when the moon and a star are observed either at the *same* altitude, or at altitudes which differ only by a quantity which can be measured with a micrometer.

243. (A.) *By the moon's absolute altitude.*—This method being practised only with portable instruments, it would be quite superfluous to employ the rigorous processes of correcting for the parallax, which require the azimuth of the moon to be given. The process of Art. 97 will, therefore, be employed in this case with advantage, by which the observed zenith distance is reduced not to the centre of the earth, but to the point of the earth's axis which lies in the vertical line of the observer, and which we briefly designate as *the point O.*   Let

$\zeta''=$ the observed zenith distance, or complement of the observed altitude, of the moon's limb,
$\Theta =$ the local sidereal time,
$L'=$ the assumed longitude,
$\Delta L =$ the required correction of L',
$L =$ the true longitude $= L' + \Delta L$.

Find the Greenwich sidereal time $\Theta + L'$, and convert it into mean time, for which take from the Ephemeris the quantities

$\delta =$ the moon's declination,
$\pi =$ "   eq. hor. parallax,
$S =$ "   semidiameter.

Let S' be the apparent semidiameter obtained by adding to S the augmentation computed by (251) or taken from Table XII. Let r be the refraction for the apparent zenith distance ζ''; and put

$$\zeta' = \zeta'' + r \pm S' \tag{434}$$

Let π_1 be the corrected parallax for the point O, found by (127), or by adding to π the correction of Table XIII. (which in the present application will never be in error $0''.1$); and put

$$\left. \begin{array}{l} \delta_1 = \delta + e^2 \pi_1 \sin \varphi \cos \delta \\ \zeta_1 = \zeta' - \pi_1 \sin \zeta' \end{array} \right\} \tag{435}$$

in which $\log e^2 = 7.8244$.

Hence, denoting the longitude computed from the right ascension $\alpha = \Theta - t$ by L'', we have

$$\text{True longitude} = L' + \Delta L = L'' - \frac{\beta \Delta L}{15\,\lambda\,\cos\delta\,\tan q}$$

whence

$$\Delta L = \frac{L'' - L}{1 + \dfrac{\beta}{15\,\lambda}\,\sec\delta\,\cot q}$$

If we denote the denominator of this expression by $1 + a$, we shall have, by (18),

$$= \frac{\beta}{15\,\lambda}\left(\frac{\tan\varphi}{\sin t} - \frac{\tan\delta}{\tan t}\right) \qquad (438)$$

and then

$$\Delta L = \frac{L'' - L'}{1 + a} \qquad L = L' + \Delta L \qquad (439)$$

EXAMPLE.—At the U. S. Naval Academy, in latitude $\varphi = 38°$ 58' 53" and assumed longitude $L' = 5^h\ 6^m\ 0^s$, I observed the double altitude of the moon's upper limb with a sextant and artificial horizon as below:

1849 May 2.—Moon east of the meridian.

Chronometer	$10^h\ 14^m\ 21^s.6$		Mean of 6 obs. $2\,\overline{\mathbb{D}}$	$=$	$64°\ 40'\ 0''$
Fast	4 41 0.0		Index corr. of sextant $=$		$-$ 14 57
Local mean time $=$	5 33 21.6				2) 64 25 3
Assumed L' $=$	5 6 0.			App. alt. $\overline{\mathbb{D}} =$	32 12 31 .5
Approx. Gr. time $=$	10 39 21.6			$\zeta'' =$	57 47 28 .5

(For which time we take π, S, and δ from the Nautical Almanac)

$\delta = + 3°\ 47'\ 47''.6$

$e^2 \pi_1 \sin\phi \cos\delta = \quad + 14\ .1$

$\delta_1 = + 3\ 48\ 1\ .7$

Barom.	$30^{in}.45$	
Att. Therm.	63° F.	$r = + 1.30\ .9$
Ext. "	65° F.	
	$S = 15'\ 16''.4$	
ΔS (Tab. XII.) $= \ + 8\ .1$		$S' = + 15\ 24\ .5$
		$\zeta' = 58\ 4\ 23\ .9$
	$\pi = 56'\ 3''.1$	
$\Delta\pi$ (Tab. XIII.) $= + 4\ .4$	$\pi_1 \sin\zeta' =$	47 38 .1
	$\pi_1 = 56\ 7\ .5$	
		$\zeta_1 = 57\ 16\ 45\ .8$

With these values of δ_1, ζ_1, and $\varphi = 38°\ 58'\ 53''$, we find, by (436),

$$t = - 3^h\ 19^m\ 53^s.64$$

The sidereal time at Greenwich mean noon, 1849 May 2, was $2^h\ 41^m\ 7^s.98$; whence

$$\Theta = \ 8^h\ 16^m\ 14^s.61$$
$$\alpha = 11\ 36\ \quad 8\ .25$$

Corresponding to this right ascension we find by the hourly Ephemeris the Greenwich mean time, and hence the longitude L'', as follows:

$$
\begin{array}{rcl}
\text{Greenwich mean time} & = & 10^h\ 39^m\ 48^s.7 \\
\text{Local} \quad `` \quad `` & = & 5\ \ 33\ \ 21\ .6 \\
\hline
L'' & = & 5\ \ \ 6\ \ 27\ .1 \\
L'' - L' & = & +\ \ 27^s.1
\end{array}
$$

By the hourly Ephemeris we also have for the Greenwich time $10^h\ 39^m\ 48^s.7$,

$$
\begin{array}{l}
\text{Increase of } \alpha \text{ in } 1^m = \lambda = +\ 2^s.014 \\
\quad `` \quad\quad \delta \text{ in } 1^m = \beta = +\ 10''.01
\end{array}
$$

and hence, by (438) and (439),

$$
\begin{array}{ll}
a = -\ 0.3317 & \Delta L = +\ 40^s.6 \\
\multicolumn{2}{c}{L = L' + \Delta L = 5^h\ 6^m\ 40^s.6}
\end{array}
$$

244. The result thus obtained involves the errors of the tabular right ascension and declination and the instrumental error. The tabular errors are removed by means of observations of the same data made at some of the principal observatories, as in the case of moon culminations. The instrumental error will be nearly eliminated by determining the local time from a star at the same altitude and as nearly as possible the same declination; for the instrumental error will then produce the same error in both Θ and t, and, therefore, will be eliminated from their difference $\Theta - t = \alpha$. The error in the longitude will then be no greater than the error in Θ. But to give complete effect to this mode of eliminating the error, an instrument, such as the zenith telescope, should be employed, which is capable of indicating the same altitude with great certainty and does not involve the errors of graduation of divided circles. A very different method of observation and computation must then be resorted to, which I proceed to consider.

245. (B.) *By equal altitudes of the moon and a star, observed with the zenith telescope.*—The reticule of this instrument should for these observations be provided with a system of fixed horizontal threads: nevertheless, we may dispense with them, and employ only the single movable micrometer thread, by setting it successively at convenient intervals.

Having selected a well determined star as nearly as possible in the moon's path and differing but little in right ascension, a preliminary computation of the approximate time when each body will arrive at some assumed altitude (not less than 10°) must be made, as well as of their approximate azimuths, in order to point the instrument properly. The instrument being pointed for the first object, the level is clamped so that the bubble plays near the middle of the tube, and is then not to be moved between the observation of the moon and the star. After the object enters the field, and before it reaches the first thread, it may be necessary to move the instrument in azimuth in order that the transits over the horizontal threads may all be observed without moving the instrument *during* these transits. The times by chronometer of the several transits are then noted, and the level is read off. The instrument is then set upon the azimuth of the second object, the observation of which is made in the same manner, and then the level is again read off. This completes one observation. The instrument may then be set for another assumed altitude, and a second observation may be taken in the same manner.* Each observation is then to be separately reduced as follows: Let

i, i', i'', &c. = the distances in arc of the several threads from their mean,

m, m' = the mean of the values of i for the observed threads, in the case of the moon and star respectively,

l, l' = the level readings, in arc, for the moon and star,

Θ, Θ' = the mean of the sidereal times of the observed transits of the moon and star;

then the excess of the observed zenith distance of the moon's limb at the time Θ above that of the star at the time Θ' is†

$$m - m' + l - l'$$

the quantities m and l being supposed to increase with increasing zenith distance.

---

* The same method of observation may be followed with the ordinary universal instrument, but, as the level is generally much smaller than that of the zenith telescope, the same degree of accuracy will not be possible.

† When the micrometer is set successively upon assumed readings, m and m' will be the means of these readings, converted into arc, with the known value of the screw.

Also, let

$a, \delta, t, \zeta, A, q =$ the R. A., decl., hour angle, geocentric zenith distance, azimuth, and parallactic angle of the moon's centre at the time Θ;

$a', \delta', t', \zeta', A', q' =$ the same for the star at the time Θ';

$\pi, S =$ the moon's equatorial hor. parallax and semidiameter;

$\lambda =$ the increase of a in 1ˢ of sid. time;

$\beta =$　　　"　　　δ　"　　"　　"

$\varphi =$ the latitude;

$L' =$ the assumed longitude;

$\Delta L =$ the required correction of L';

The quantities $a, \delta, \pi,$ and S are to be taken from the Ephemeris for the Greenwich sidereal time $\Theta + L'$ (converted into mean time); a and δ being interpolated with second differences by the hourly Ephemeris. Then the required correction of the longitude will be found by comparing the computed value of ζ with the observed value. For this purpose we first compute ζ and ζ' with the greatest precision. and also A and q approximately. If the differential formula of the next article is also to be computed, A' and q' will also be required. The most convenient formulæ will be—

For the moon.

$$t = \Theta - a$$

$$\left.\begin{array}{l} \tan M = \tan \delta \sec t \\ \cos \zeta = \dfrac{\sin \delta \cos (\varphi - M)}{\sin M} \end{array}\right\} \text{with six decimals;}$$

For the star.

$$\left\{\begin{array}{l} t' = \Theta' - a' \\ \tan M' = \tan \delta' \sec t' \\ \cos \zeta' = \dfrac{\sin \delta' \cos (\varphi - M')}{\sin M'} \end{array}\right.$$

$$\left.\begin{array}{l} \cos A = \tan (\varphi - M) \cot \zeta \\ \tan N = \cot \varphi \cos t \\ \tan q = \dfrac{\tan t \sin N}{\cos (\delta + N)} \end{array}\right\} \text{with four decimals;}$$

$$\left\{\begin{array}{l} \cos A' = \tan (\varphi - M') \cot \zeta' \\ \tan N' = \cot \varphi \cos t' \\ \tan q' = \dfrac{\tan t' \sin N'}{\cos (\delta' + N')} \end{array}\right.$$

$$(440)$$

The zenith distance ζ thus computed will not strictly correspond to the time Θ unless the assumed longitude is correct. Let its true value be $\zeta + d\zeta$. Also put

$\zeta_1 =$ the observed zenith distance of the moon's limb,

$\zeta_1' =$ the observed zenith distance of the star,

$r, r' =$ the refraction for ζ, and ζ',

then

$$\zeta_1' = \zeta' - r'$$
$$\zeta_1 = \zeta_1' + m - m' + l - l'$$

Putting then

$$\zeta'' = \zeta_1 + r = \zeta' + m - m' + l - l' + (r - r')$$

and, by Art. (136),

$$\gamma = (\varphi - \varphi') \cos A \qquad \sin p = \rho \sin \pi \sin (\zeta'' - \gamma)$$

$$k = p \mp S \mp \tfrac{1}{2}(p \mp S) \sin p \sin S$$

$$\left.\begin{array}{}\end{array}\right\} \quad (441)$$

the $\left\{\begin{array}{c}\text{upper}\\\text{lower}\end{array}\right\}$ sign being used for the moon's $\left\{\begin{array}{c}\text{upper}\\\text{lower}\end{array}\right\}$ limb, we have

$$\zeta'' - (\zeta + d\zeta) = k$$

This equation determines $d\zeta$. We have, therefore, only to determine the relation between $d\zeta$ and ΔL. Now, we have taken α and δ for the Greenwich sidereal time $\Theta + L'$, when we should have taken them for the time $\Theta + L' + \Delta L'$: hence

$$\begin{array}{lll} \alpha \text{ requires the correction} & \lambda \Delta L \\ \delta \quad\text{``} & \text{``} & \beta \Delta L \\ t \quad\text{``} & \text{``} & -\lambda \Delta L \end{array}$$

and then, by (51),

$$d\zeta = -\cos q \cdot \beta \Delta L - \sin q \cos \delta \cdot 15 \lambda \Delta L$$

Hence, putting $x = -d\zeta$, or

$$x = \zeta - \zeta'' + k$$

and

$$a = 15 \lambda \sin q \cos \delta + \beta \cos q$$

we have

$$\Delta L = \frac{x}{a} \qquad L = L' + \Delta L$$

$$\left.\begin{array}{}\end{array}\right\} \quad (442)$$

The solution of the problem, upon the supposition that all the data are correct, is completely expressed by the equations (440), (441), and (442).

246. The quantity x is in fact produced not only by the error in the assumed longitude, but also by the errors of observation and of the Ephemeris. In order to obtain a general expression

in which the effect of every source of error may be represented, let

$$T, T' = \text{the chronometer times of observation of the moon and star,}$$
$$\Delta T = \text{the assumed chronometer correction,}$$
$$\delta T, \delta T' = \text{the corrections of } T \text{ and } T' \text{ for errors of observation,}$$
$$\delta \Delta T = \text{the correction of } \Delta T,$$
$$\delta a, \delta \delta, \delta \pi, \delta S = \text{the corrections of the elements taken from the Ephemeris,}$$
$$\delta \varphi = \text{the correction of the assumed latitude.}$$

If, when the corrected values of all the elements are substituted, ζ, ζ', k become $\zeta + d\zeta$, $\zeta' + d\zeta'$, $k + dk$, instead of the equation $\zeta'' - (\zeta + d\zeta) = k$ we shall have

$$\zeta'' + d\zeta' - (\zeta + d\zeta) = k + dk$$

and hence

$$x = - d\zeta + d\zeta' - dk \qquad (443)$$

and we have now to find expressions for $d\zeta$, $d\zeta'$, and dk in terms of the above corrections of the elements.

Taking all the quantities as variables, we have

$$d\zeta = 15 \sin q \, \cos \delta \, dt - \cos q \, d\delta + \cos A \, d\varphi$$
$$d\zeta' = 15 \sin q' \cos \delta' \, dt' - \cos q' \, d\delta' + \cos A' \, d\varphi$$

Since $t = T + \Delta T - \alpha$, we have

$$dt = dT + d\Delta T - d\alpha$$

where dT and $d\Delta T$ may be exchanged for δT and $\delta \Delta T$, but $d\alpha$ is composed of two parts: 1st, of the actual correction of the Ephemeris; and 2d, of $\lambda(\Delta L + \delta T + \delta \Delta T)$ resulting from our having taken α for the uncorrected time: hence we have

$$dt = dT + \delta \Delta T - \delta \alpha - \lambda(\Delta L + \delta T + \delta \Delta T)$$

The correction $d\delta$ is also composed of two parts, so that

$$d\delta = \delta \delta + \beta(\Delta L + \delta T + \delta \Delta T)$$

Further, we have simply $d\delta' = \delta \delta'$, and

$$dt' = \delta T' + \delta \Delta T - \delta \alpha'$$

in which $\delta \Delta T$ at the time T' is assumed to be the same as at the

time T, an error in the rate of chronometer being insensible in the brief interval between the observations of the moon and the star.

Again, we have, from (441),

$$\cos p \, dp = \rho \cos \pi \sin (\zeta'' - \gamma) \, d\pi + \rho \sin \pi \cos (\zeta'' - \gamma) \, d\zeta''$$
$$dk = dp \mp dS$$

or, with sufficient accuracy,

$$dk = \sin \zeta' \, \delta\pi \mp \delta S + \sin \pi \cos \zeta' \, d\zeta'$$

Now, substituting in $d\zeta$ and $d\zeta'$ the values of dl, $d\delta$, &c., and then substituting the values of $d\zeta$ and $d\zeta'$ thus found, in (443), together with the value of dk, we obtain the final equation desired, which may be written as follows:*

$$\left.\begin{aligned}
x = {}& a \Delta L + f \cdot \delta a + \cos q \cdot \delta\delta - (f - a) \, \delta T \\
& - mf' \cdot \delta a' - m \cos q' \, \delta\delta' + mf' \cdot \delta T' \\
& \pm \delta S - \sin \zeta' \, \delta\pi - (f - mf' - a) \, \delta \Delta T \\
& - (\cos A - m \cos A') \, \delta\varphi
\end{aligned}\right\} \quad (444)$$

where the following abbreviations are employed:

$$f = 15 \sin q \cos \delta \qquad\qquad f' = 15 \sin q' \cos \delta'$$
$$a = \lambda f + \beta \cos q \qquad\qquad m = 1 - \sin \pi \cos \zeta'$$

Having computed the equation in this form, every term is to be divided by a, and then ΔL will be obtained in terms of x and all the corrections of the elements.

A discussion of this equation, quite similar to that of (433), will readily show that the observations will give the best result when taken near the prime vertical and in low latitudes, and, farther, that the combination of observations equidistant from the meridian, east and west, eliminates almost wholly errors of declination and parallax and of the chronometer correction.

EXAMPLE.†—At Batavia, on the 11th of October, 1853, Mr. DE LANGE, among other observations of the same kind, noted the following times by a sidereal chronometer, when the moon's

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* The formula (444) is essentially the same as that given by OUDEMANS, *Astronom. Journal*, Vol. IV. p. 164. The method itself is the suggestion of Professor KAISER of the Netherlands.

† *Astronomical Journal*, Vol. IV. p. 165.

lower limb and 36 *Capricorni* passed the same fixed horizontal threads:

$$T = 0^h\ 38^m\ 8^s.62 \qquad\qquad T' = 0^h\ 49^m\ 53^s.77$$

The difference of the zenith distances indicated by the level was

$$l - l' = + 2''.0$$

The chronometer correction was $\Delta T = + 1^m\ 3^s.32$, and the rate in the interval $T' - T$ was insensible.

$$\text{The assumed latitude was } \varphi = -\ 6°\ 9'\ 57''.0$$
$$\text{``}\qquad \text{longitude ``}\quad L' = -\ 7^h\ 7^m\ 37^s.0$$

We have

$$\Theta = 0^h\ 39^m\ 11^s.94 \qquad\qquad \Theta' = 0^h\ 50^m\ 57^s.09$$

For the Greenwich sid. time $\Theta + L' = 17^h\ 31^m\ 34^s.94$, or mean time $4^h\ 10^m\ 57^s.00$, we find, from the Nautical Almanac,

$$a = \quad 21^h\ 12^m\ 5^s.45 \qquad\qquad \lambda = +\ 0^s.0387$$
$$\delta = -\ 20°\ 55'\ 8''.9 \qquad\qquad \beta = +\ 0''.1440$$
$$\pi = \quad 57'\ 51''.4 \qquad\qquad a' = \quad 21^h\ 20^m\ 22^s.45$$
$$S = \quad 15'\ 47''.8 \qquad\qquad \delta' = -\ 22°\ 26'\ 30''.5$$

The computation by (440) gives

$$\zeta = 52°\ 11'\ 49''.44 \qquad\qquad \zeta' = 53°\ 13'\ 57''.30$$
$$A = 68°\ 14'.4 \qquad\qquad A' = 66°\ 30'.6$$
$$q = 81°\ 18'.9 \qquad\qquad q' = 80°\ 35'.2$$

From Table III. we find

$$\varphi - \varphi' = -\ 2'\ 27'' \qquad\qquad \log \rho = 9.999983$$

Since the same fixed threads were used for both moon and star, we have $m = m'$, and hence also sensibly $r = r'$; therefore, by (441), we find

$$\zeta'' = 53°\ 13'\ 59''.30 \qquad \gamma = -\ 54''.5 \qquad p = 46'\ 21''.25$$
$$\zeta - \zeta'' = -\ 62'\ 9''.86 \qquad k = +\ 62'\ 9''.17$$

Hence, by (442),

$$x = -\ 0''.69 \qquad\qquad a = +\ 0.5575 \qquad\qquad \Delta L = -\ 1^s.24$$

The longitude by this observation, if the Ephemeris is correct, is therefore

$$L = L' + \Delta L = -\ 7^h\ 7^m\ 38^s.24$$

If we compute all the terms of (444), we shall find

$$\Delta L = - 1^{s}.24 - 24.84\,\delta a - 0.27\,\delta\delta + 23.84\,\delta T - 24.24\,\delta T' - 0.44\,\delta_\Delta T$$
$$+ 24.28\,\delta a' + 0.29\,\delta\delta' + 1.79\,\delta S + 1.44\,\delta\pi - 0.04\,\delta\varphi$$

This shows clearly the effect of each source of error; but in practice it will usually be sufficient to compute only the coefficients of δa and $\delta\delta$. In the present example, therefore, we should take

$$\Delta L = - 1^{s}.24 - 24.84\,\delta a - 0.27\,\delta\delta$$

which will finally be fully determined when δa and $\delta\delta$ have been found from nearly corresponding observations at Greenwich or elsewhere.

SEVENTH METHOD.—BY LUNAR DISTANCES.

247. The distance of the moon from a star may be employed in the same manner as the right ascension was employed in Arts. 229, &c., to determine the Greenwich time, and hence the longitude. If the star lies directly in the moon's path, the change of distance will be even more rapid than the change of right ascension; and therefore if the distance could be measured with the same degree of accuracy as the right ascension, it would give a more accurate determination of the Greenwich time. The distance, however, is observed with a sextant, or other reflecting instrument (see Vol. II.), which being usually held in the hand is necessarily of small dimensions and relatively inferior accuracy. Nevertheless, this method is of the greatest importance to the travelling astronomer, and especially to the navigator, as the observation is not only extremely simple and requires no preparation, but may be practised at almost any time when the moon is visible.

The Ephemerides, therefore, give the true distance of the centre of the moon from the sun, from the brightest planets, and from nine bright fixed stars, selected in the path of the moon, for every third hour of mean Greenwich time. The planets employed are Saturn, Jupiter, Mars, and Venus. The nine stars, known as lunar-distance stars, are α Arietis, α Tauri (*Aldebaran*), β Geminorum (*Pollux*), α Leonis (*Regulus*), α Virginis (*Spica*), α Scorpii (*Antares*), α Aquilæ (*Altair*), α Piscis Australis (*Fomalhaut*), and α Pegasi (*Markab*).

The distance *observed* is that of the moon's bright limb from a

by refraction if we wish to compute the true distance to the nearest second of arc (Art. 133). These features, which add very materially to the labor of computation, cannot be overlooked in any complete discussion of the problem.

Simple as the problem appears when stated generally, the strict computation of it is by no means brief; and its importance and the frequency of its application at sea, where long computations are not in favor, have led to numerous attempts to abridge it. In most instances the abbreviations have been made at the expense of precision; but in the methods given below the error in the computation will always be much less than the probable error of the best observation with reflecting instruments: so that these methods are entitled to be considered as practically perfect.

With the single exception of that proposed by BESSEL,[*] all the solutions depend upon the two triangles of Fig. 29, and may be divided into two classes, *rigorous* and *approximative*. In the rigorous methods the true distance is directly deduced by the rigorous formulæ of Spherical Trigonometry; but in the approximative methods the difference between the apparent and the true distance is deduced either by successive approximations or from a development in series of which the smaller terms are neglected. Practically, the latter may be quite as correct as the former, and, indeed, with the same amount of labor, more correct, since they require the use of less extended tables of logarithms. I propose to give two methods, one from each of these classes.

A.—*The Rigorous Method.*

248. For brevity, I shall call the body from which the moon's distance is observed *the sun*, for our formulæ will be the same for a planet, and for a fixed star they will require no other change than making the parallax and semidiameter of the star zero.

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[*] *Astron. Nach.* Vol. X. No. 218, and *Astron. Untersuchungen*, Vol. II. BESSEL'S method requires a different form of lunar Ephemeris from that adopted in our Nautical Almanacs. But even with the Ephemeris arranged as he proposes, the computation is not so brief as the approximative method here given, and its superiority in respect of precision is so slight as to give it no important practical advantage. It is, however, the only theoretically *exact* solution that has been given, and might still come into use if the *measurement* of the distance could be rendered much more precise than is now possible with instruments of reflection.

Let us suppose that at the given local mean time T the observation (or, in the case of the altitudes, computation) has given

$d''=$ the apparent distance of the limbs of the moon and sun,

$h'=$ the apparent altitude of the moon's centre,

$H'=$ the apparent altitude of the sun's centre,

and that in order to compute the refraction accurately the barometer and thermometer have also been observed. For the Greenwich time corresponding to T, which will be found with sufficient accuracy for the purpose by employing the supposed longitude, take from the Ephemeris

$s=$ the moon's semidiameter,

$S=$ the sun's   "

then, putting

$d'=$ the apparent distance of the centres,

$s'=$ the moon's augmented semidiameter,

$=s+$ correction of Table XII.

we have

$$d'=d''\pm s'\pm S$$

upper signs for nearest (inner) limbs, lower signs for farthest (outer) limbs.

But if the altitude of either body is less than 50°, we must take into account the elliptical figure of the disc produced by refraction. For this purpose we must employ, instead of s' and S, those semidiameters which lie in the direction of the lunar distance. Putting

$$q=ZM'S', \qquad\qquad Q=ZS'M' \text{ (Fig. 29)}$$

$\Delta s, \Delta S=$ the contraction of the *vertical* semidiameters of the moon and sun for the altitudes h' and H',

the required inclined semidiameters will be (Art. 133)

$$s'-\Delta s\cos^2 q \qquad \text{and} \qquad S-\Delta S\cos^2 Q$$

The angles q and Q will be found from the three sides of the triangle $ZM'S'$, taking for d' its approximate value $d''\pm s'\pm S$ (which is sufficiently exact for this purpose, as great precision in q and Q is not required), and for the other sides $90°-h'$ and $90°-H'$. If we put

$$m=\tfrac{1}{2}(h'+H'+d')$$

we shall have

$$\sin^2 \tfrac{1}{2} q = \frac{\cos m \sin(m - H')}{\sin d' \cos h'} \qquad \sin^2 \tfrac{1}{2} Q = \frac{\cos m \sin(m - h')}{\sin d' \cos h'} \quad (445)$$

and then the apparent distance by the formula

$$d' = d'' \pm (s' - \Delta s \cos^2 q) \pm (S - \Delta S \cos^2 Q) \qquad (446)$$

We are now to reduce the distance to the centre of the earth. We shall first reduce it to that point of the earth's axis which lies in the vertical line of the observer. Designating this point as *the point O*, Art. 97, let

$d_1, h_1, H_1 =$ the distance and altitudes reduced to the point O,

$r, R =$ the refraction for the altitudes h' and H',

$\pi, P =$ the equatorial hor. parallax of the moon and sun.

The moon's parallax for the point O will be found rigorously by (127), but with even more than sufficient precision for the present problem by adding to π the correction given by Table XIII. Denoting this correction by $\Delta\pi$, we have

$$\pi_1 = \pi + \Delta\pi$$

$$h_1 = h' - r + \pi_1 \cos(h' - r) \qquad H_1 = H' - R + P \cos(H' - R) \quad (447)$$

The parallax P is in all cases so small that its reduction to the point O is insignificant.

If, then, in Fig. 29, M and S represent the moon's and sun's places reduced to the point O, and we put

$$Z = \text{the angle at the zenith, } MZS,$$

we shall have given in the triangle $M'ZS'$ the three sides d', $90° - h'$, $90° - H'$, whence

$$\cos^2 \tfrac{1}{2} Z = \frac{\cos \tfrac{1}{2}(h' + H' + d') \cos \tfrac{1}{2}(h' + H' - d')}{\cos h' \cos H'}$$

and, then, in the triangle MZS we shall have given the angle Z with the sides $90° - h_1$ and $90° - H_1$, whence the side $MS = d_1$ will be found by the formula [Sph. Trig. (17)],

$$\sin^2 \tfrac{1}{2} d_1 = \cos^2 \tfrac{1}{2}(h_1 + H_1) - \cos h_1 \cos H_1 \cos^2 \tfrac{1}{2} Z$$

To simplify the computation, put

$$m = \tfrac{1}{2}(h' + H' + d')$$

then the last formula, after substituting the value of Z, becomes,

$$\sin^2 \tfrac{1}{2} d_1 = \cos^2 \tfrac{1}{2}(h_1 + H_1) - \frac{\cos h_1 \cos H_1}{\cos h' \cos H'} \cos m \cos(m - d')$$

Let the auxiliary angle M be determined by the equation

$$\sin^2 M = \frac{\cos h_1 \cos H_1}{\cos h' \cos H'} \cdot \frac{\cos m \cos(m - d')}{\cos^2 \tfrac{1}{2}(h_1 + H_1)} \qquad (448)$$

then we have*

$$\sin \tfrac{1}{2} d_1 = \cos \tfrac{1}{2}(h_1 + H_1) \cos M \qquad (449)$$

Finally, to reduce the distance from the point O to the centre of the earth, let P (Fig. 30) be the north pole of the heavens, M_1 the moon's place as seen from the point O, M the moon's geocentric place, S the sun's place (which is sensibly the same for either point). The point O being in the axis of the celestial sphere, the points M_1 and M evidently lie in the same declination circle PM_1M. Hence, putting

Fig. 30.
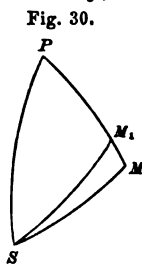
$d =$ the geocentric distance of the moon and sun $= SM$,
$d_1 = SM_1$,
$\delta =$ the moon's geocentric declination $= 90° - PM$,
$\delta_1 =$ the declination reduced to the point $O = 90° - PM_1$,
$\varDelta =$ the sun's declination,

we have, in the triangles PMS and M_1MS,

$$\cos PMS = \frac{\cos d_1 - \cos(\delta_1 - \delta) \cos d}{\sin(\delta_1 - \delta) \sin d} = \frac{\sin \varDelta - \sin \delta \cos d}{\cos \delta \sin d}$$

We may put $\cos(\delta_1 - \delta) = 1$, and, therefore,

$$\cos d_1 - \cos d = \frac{\sin(\delta_1 - \delta)}{\cos \delta}(\sin \varDelta - \sin \delta \cos d)$$

---

* This transformation of the formulæ is due to Borda, *Description et usage du cercle de réflexion.*

Then to compute the contraction produced by refraction we find from the refraction table, for the given observed altitudes, the contractions of the *vertical* semidiameters (Art. 132),

$$\Delta s = 0''.4 \qquad \Delta S = 9''.6$$

With the approximate altitudes and distance of the centres we then proceed by (445), as follows:

$d' = 45° 10'$	log cosec d'	0.1493		log cosec d'	0.1493
$h' = 52\ 51$	log sec h'	0.2190			
$H' = \ \ 9\ 12$				log sec H'	0.0056
$m = 53\ 37$	log cos m	9.7782		log cos m	9.7732
$m - H' = 44\ 25$	log sin $(m - H')$	9.8450			
$m - h' = \ \ 0\ 46$				log sin $(m - h')$	8.1265
					8.0546
		9.9865			
	log sin $\frac{1}{2} q$	9.9933		log sin $\frac{1}{2} Q$	9.0273
	$q =$	159° 56'		$Q =$	12° 14'
	log cos² q	9.9456		log cos² Q	9.9800
	log Δs	9.6021		log ΔS	0.9823
		9.5477			0.9623
	Δs cos² $q =$	0.''4		ΔS cos² $Q =$	9''.2

Hence we have, by (446),

$$d'' = 44° 36'\ 58''.6$$
$$s' - \Delta s\ \cos^2 q = \qquad 16\ 36\ .7$$
$$S - \Delta S\ \cos^2 Q = \qquad 15\ 58\ .8$$
$$d' = 45 \quad 9\ 34\ .1$$

3d. To find the apparent and true altitudes of the centres.—The apparent altitudes of the centres will be found by adding the contracted vertical semidiameters to the observed altitudes of the limbs. The apparent altitudes, however, need not be computed with extreme precision, provided that the differences between them and the true altitudes are correct; for it is mainly upon these differences that the difference between the apparent and true distance depends.

The reduction of the moon's horizontal parallax to the point O for the latitude 35° is, by Table XIII., $\Delta \pi = 3''.9$; and hence we have

$$\pi_1 = \pi + \Delta \pi = 60'\ 5''.8$$

and the computation of the altitudes by (447) is as follows:

$$h'' = 52° 34' \ 0'' \qquad\qquad H'' = 8° 56' 23''$$

$h'' =$	$52° 34'$	$0''$	$H'' =$	$8° 56' 23''$
Vert. semid. $=$	16	37	Vert. semid. $=$	$15 \ 58$
$h' =$	$52 \ 50$	37	$H' =$	$9 \ 12 \ 21$
Table I. $r =$		$42 \ .7$	$R =$	$5 \ 33 \ .6$
$h' - r =$	$52 \ 49$	$54 \ .3$	$H' - R \ =9$	$6 \ 47 \ .4$

$\log \pi_1$	3.55700	$\log P$	0.9345
$\log \cos (h' - r)$	9.78115	$\log \cos (H' - R)$	9.9945
	3.33815		0.9290
$\pi_1 \cos (h' - r) =$	$36' \ 18''.5$	$P \cos (H' - R) =$	$8''.5$
$h_1 =$	$53° 26' \ 12''.8$	$H_1 =$	$9° 6' \ 55''.9$

4th. We now find the distance d_1 by (448) and (449), as follows:

$d' =$	$45° \ 9' \ 34''.1$		
$h' =$	$52 \ 50 \ 37$	$\log \sec$	0.2189683
$H' =$	$9 \ 12 \ 21$	$\log \sec$	0.0056335
$m =$	$53 \ 36 \ 16 \ .1$	$\log \cos$	9.7733154
$m - d' =$	$8 \ 26 \ 42 \ .$	$\log \cos$	9.9952654
$h_1 =$	$53 \ 26 \ 12 \ .8$	$\log \cos$	9.7750333
$H_1 =$	$9 \ 6 \ 55 \ .9$	$\log \cos$	9.9944803
		$2)$	9.7626962
			9.8813481
$\frac{1}{2}(h_1 + H_1) =$	$31 \ 16 \ 34 \ .4$	$\log \cos$	9.9318007 .... 9.9318007
		$\log \sin M$	9.9495474 $\quad \log \cos M \ 9.6583265$
$\frac{1}{2} d_1 =$	$22 \ 54 \ 7 \ .9$		$\log \sin \frac{1}{2} d_1 \ 9.5901272$
$d_1 =$	$45 \ 48 \ 15 \ .8$		

5th. To find the geocentric distance, we have, by (450), for $\varphi = 35°$,

$\log A$	7.8249	$\delta = + 14° 19'$	
$\log \pi$	3.5565	$\varDelta = - 4 \ 3$	
$\log \sin \varphi$	9.7586		
	1.1400 .... 1.1400		
$\log \sin \varDelta$	$n8.8490$	$\log \sin \delta$	9.3932
$\log \cosec d_1$	0.1445	$\log \cot d_1$	0.0122
	$n0.1335$		$n0.5454$
	$- 1''.4$		$- 3''.5$
	$d - d_1 = - 4''.9$		
	$d = 45° 48' 10''.9$		

6th. To find the Greenwich mean time corresponding to d,

and hence the longitude, according to Art. 66, we find an approximate time $(T) + t$ by simple interpolation, and then the required time $T_0 = (T) + t + \Delta t$, taking Δt from Table XX., with the arguments t and ΔQ (= increase of the logarithms in the Ephemeris in 3^h), as follows:

By the American Ephemeris of 1856 for March 9, we have

$$(T) = 15^h\ 0^m\ 0^s \quad (d) = 45°\ 40'\ 54'' \qquad Q = 0.2510 \quad \Delta Q = +17$$
$$d = 45\ \ 48\ 10.9$$

$$t = 0\ 12\ 59 \qquad\qquad\quad 7\ 16.9 \quad \log = 2.6404$$
$$\Delta t = -\ \ 1 \qquad\qquad\qquad\qquad\qquad \log t = 2.8914$$
$$T_0 = 15\ 12\ 58$$
$$T = 5\ 14\ 6$$
$$L = 9\ 58\ 52$$

B.—*The Approximative Method.*

249. I shall here give my own method (first published in the *Astronomical Journal*, Vol. II.), as it yet appears to me to be the shortest and most simple of the approximative methods *when these are rendered sufficiently accurate by the introduction of all the necessary corrections.* Its value must be decided by the importance attached to a precise result. There are briefer methods to be found in every work on Navigation, which will (and should) be preferred in cases where only a rude approximation to the longitude is required.

As before, let

h', H' = the apparent altitudes of the centres of the moon and sun,

d'' = the observed distance of the limbs,

s, S = their geocentric semidiameters,

π, P = their equatorial horizontal parallaxes,

s' = the moon's semidiameter, augmented by Table XII.,

π_1 = the moon's parallax, augmented by Table XIII.

We shall here also first reduce the distance to the point O of Art. 97. The contractions of the semidiameters produced by refraction will be at first disregarded, and a correction on that account will be subsequently investigated. If then in Fig. 29, p. 394, M' and S' denote the apparent places, M and S the places reduced to the point O, we shall here have

$$d' = d'' \pm s' \pm S = M'S', \qquad d_1 = MS,$$
$$h' = 90° - ZM', \qquad H' = 90° - ZS',$$
$$h_1 = 90° - ZM, \qquad H_1 = 90° - ZS,$$

and the two triangles give

$$\cos Z = \frac{\cos d_1 - \sin h_1 \sin H_1}{\cos h_1 \cos H_1} = \frac{\cos d' - \sin h' \sin H'}{\cos h' \cos H'}$$

from which, if we put

$$m = \frac{\sin h_1 \sin H_1}{\sin h' \sin H'} \qquad n = \frac{\cos h_1 \cos H_1}{\cos h' \cos H'}$$

we derive

$$\cos d' - \cos d_1 = (1 - n) \cos d' + (n - m) \sin h' \sin H' \qquad (a)$$

Put

$$\Delta d = d_1 - d' \qquad \Delta h = h_1 - h' \qquad \Delta H = H' - H_1 \qquad (b)$$

then we have

$$\cos d' - \cos d_1 = 2 \sin \tfrac{1}{2} \Delta d \sin (d' + \tfrac{1}{2} \Delta d) \qquad (c)$$

and

$$n = \frac{\cos (h' + \Delta h)}{\cos h'} \cdot \frac{\cos (H' - \Delta H)}{\cos H'}$$

$$= \left(1 - \frac{2 \sin \tfrac{1}{2} \Delta h \sin (h' + \tfrac{1}{2} \Delta h)}{\cos h'} \right) \times \left(1 + \frac{2 \sin \tfrac{1}{2} \Delta h \sin (H' - \tfrac{1}{2} \Delta H)}{\cos H'} \right)$$

$$1 - n = \frac{2 \sin \tfrac{1}{2} \Delta h \sin (h' + \tfrac{1}{2} \Delta h)}{\cos h'} - \frac{2 \sin \tfrac{1}{2} \Delta H \sin (H' - \tfrac{1}{2} \Delta H)}{\cos H'}$$

$$+ \frac{4 \sin \tfrac{1}{2} \Delta h \sin \tfrac{1}{2} \Delta H \sin (h' + \tfrac{1}{2} \Delta h) \sin (H' - \tfrac{1}{2} \Delta H)}{\cos h' \cos H'} \qquad (d)$$

Also

$$n - m = \frac{\sin h' \cos h_1 \sin H' \cos H_1 - \cos h' \sin h_1 \cos H' \sin H_1}{\sin h' \cos h' \sin H' \cos H'}$$

substituting in which the values

$$2 \sin h' \cos h_1 = \sin (2 h' + \Delta h) - \sin \Delta h$$
$$2 \cos h' \sin h_1 = \sin (2 h' + \Delta h) + \sin \Delta h$$
$$2 \sin H' \cos H_1 = \sin (2 H' - \Delta H) + \sin \Delta H$$
$$2 \cos H' \sin H_1 = \sin (2 H' - \Delta H) - \sin \Delta H$$

we find

$$n - m = \frac{\sin \Delta H \sin (2 h' + \Delta h) - \sin \Delta h \sin (2 H' - \Delta H)}{2 \sin h' \cos h' \sin H' \cos H'} \qquad (e)$$

Substituting (c), (d), and (e) in (a), and at the same time, for brevity, putting

$$A_1 = \frac{2 \sin \frac{1}{2} \Delta h \sin (h' + \frac{1}{2} \Delta h)}{\cos h'}$$

$$B_1 = -\frac{\sin \Delta h \sin (2 H' - \Delta H)}{2 \cos h' \cos H'}$$

$$C_1 = -\frac{2 \sin \frac{1}{2} \Delta H \sin (H' - \frac{1}{2} \Delta H)}{\cos H'}$$

$$D_1 = \frac{\sin \Delta H \sin (2 h' + \Delta h)}{2 \cos h' \cos H'}$$

we have

$$2 \sin \tfrac{1}{2} \Delta d \sin (d' + \tfrac{1}{2} \Delta d) = A_1 \cos d' + B_1 + C_1 \cos d' + D_1 - A_1 C_1 \cos d' \quad (f)$$

This formula is rigorously exact; but, since Δd is always less than $1°$, it will not produce an error of $0''.1$ to substitute the arcs $\frac{1}{2} \Delta d$, $\frac{1}{2} \Delta h$, &c. for their sines, or $\frac{1}{2} \Delta d \sin 1''$, $\frac{1}{2} \Delta h \sin 1''$, &c. for $\sin \frac{1}{2} \Delta d$, $\sin \frac{1}{2} \Delta h$, &c.; and therefore we may write

$$\Delta d \sin (d' + \tfrac{1}{2} \Delta d) = A_1 \cos d' + B_1 + C_1 \cos d' + D_1 - A_1 C_1 \sin 1'' \cos d' \quad (g)$$

in which A_1, B_1, C_1, D_1, now have the following signification:

$$A_1 = \frac{\Delta h}{\cos h'} \cdot \sin (h' + \tfrac{1}{2} \Delta h)$$

$$B_1 = -\frac{\Delta h}{\cos h'} \cdot \frac{\sin (2 H' - \Delta H)}{2 \cos H'}$$

$$C_1 = -\frac{\Delta H}{\cos H'} \cdot \sin (H' - \tfrac{1}{2} \Delta H)$$

$$D_1 = \frac{\Delta H}{\cos H'} \cdot \frac{\sin (2 h' + \Delta h)}{2 \cos h'}$$

The next step in our transformation consists in finding convenient and at the same time sufficiently accurate expressions of Δh and ΔH. Let

r, $R =$ the true refractions for the apparent altitudes h' and H';

then we have, within less than $0''.1$,

$$\Delta h = \pi_1 \cos (h' - r) - r$$

If we neglect r in the term $\pi_1 \cos (h' - r)$, the error in this term will never exceed $1''$; but even this error will be avoided by taking the approximate expression

$$\cos (h' - r) = \cos h' + \sin r \sin h'$$

and we shall then have

$$\Delta h = \pi_1 \cos h' - r + \pi_1 \sin r \sin h'$$
$$= (\pi_1 \cos h' - r)\left(1 + \frac{\pi_1 \sin r \sin h'}{\pi_1 \cos h' - r}\right)$$

Since the second term of the second factor produces but $1''$ in Δh, we may employ for it an approximate value, which will still give Δh with great precision. Denoting this term by k, we have

$$k = \frac{\pi_1 \sin r \sin h'}{\pi_1 \cos h' - r} = \frac{\sin r \tan h'}{1 - \dfrac{r}{\pi_1 \cos h'}}$$

or, very nearly,

$$k = \sin r \tan h'\left(1 + \frac{r}{\pi_1 \cos h'}\right)$$

If we put

$$r = a \cot h',$$

in which a has the value given in Table II., we have

$$k = a \sin 1''\left(1 + \frac{a}{\pi_1 \sin h'}\right)$$

Now, a increases with h', but in such a ratio that k remains very nearly constant for a constant value of π_1. We may without sensible error take $\pi_1 = 57' 30'' = 3450''$, which is about the mean value of π_1, and we shall find for a mean state of the air, by the values of a given in Table II.,

$$\begin{array}{ll} \text{for } h' = 5° & k = 0.000291 \\ h' = 45 & k = 0.000286 \\ h' = 90 & k = 0.000285 \end{array}$$

Hence, if we take

$$k = 0.00029$$

the formula

$$\Delta h = (\pi_1 \cos h' - r)(1 + k) \tag{452}$$

will give Δh within $\frac{1}{200000}$ of its whole amount, that is, within less than $0''.02$ in a mean state of the air. For extreme variations

of the density of the air, it is possible that the refraction may be increased by its one-sixth part, and k will also be increased by its one-sixth part. But, as the term depending on k is not more than $1''$, the error in Δh, even in the improbable case supposed, will not be greater than $0''.16$. The formula (452) may therefore be regarded as practically exact with the value $k = 0.00029$.

A strict computation of the sun's or a planet's altitude requires the formula

$$\Delta H = R - P \cos (H' - R)$$

but P is in all cases so small that the formula

$$\Delta H = R - P \cos H' \tag{453}$$

will always be correct within a very small fraction of a second.

Now, let

$$r' = \frac{r}{\cos h'} \qquad\qquad R' = \frac{R}{\cos H'} \tag{454}$$

The quantities r' and R' computed from the mean values of the refraction are given in Table XIV. under the name "Mean Reduced Refraction for Lunars." The numbers of the table are corrected for the height of the barometer and thermometer by means of Table XIV.A and B. These tables are computed from BESSEL's refraction table, assuming the attached thermometer of the barometer, and the external thermometer, to indicate.the same temperature, which is allowable in our present problem.* By the introduction of r' and R', we obtain

$$\frac{\Delta h}{\cos h'} = (\pi_1 - r')(1 + k) \qquad\qquad \frac{\Delta H}{\cos H'} = R' - P$$

and the coefficients of formula (g) become

---

* If it is desired to compute r' and R' with the utmost rigor, it can be done by Table II., by taking (Art. 107)

$$r' = \frac{\alpha \beta^A \gamma^\lambda}{\sin h'} \qquad\qquad R' = \frac{\alpha \beta^A \gamma^\lambda}{\sin H'}$$

The tables XV. and XV.A and B give the correct values to the nearest second in all practical cases.

so that if we put

$$x = -\frac{\Delta d^2 \sin 1'' \cos (d' + \frac{1}{3}\Delta d)}{2 \sin d'}$$

or, within $0''.15$,

$$x = -\frac{1}{3}\Delta d^2 \sin 1'' \cot d' \qquad (457)$$

we have

$$\Delta d = A' + B' + C' + D' + x \qquad (458)$$

The terms A', B', C', and D' are computed directly from the apparent distance and altitudes by (456), and with sufficient accuracy with four-figure logarithms. The logarithms of A, B, C, D, are given in Table XV., $\log A$ and $\log D$ with the arguments $\pi_1 - r'$ and h'; $\log B$ and $\log C$ with the arguments $R' - P$ and H'. In the construction of this table Δh and ΔH are computed by (452) and (453), and then the logarithms of A, B, C, D, by (455).

The sum $A' + B' + C' + D'$ is called the "first correction of the distance," and, being very nearly equal to Δd, is used as the argument of Table XVI., which gives x, or the "second correction of the distance," computed by (457). When x is greater than $30''$ and the distance small, it will be necessary to enter this table a second time with the more correct value of Δd found by employing the first value of x.

The correction Δd being thus found and added to d', we have d_1, or the distance reduced to the point O. The reduction to the centre of the earth is then made by (450). This reduction is also facilitated by a table. If we put

$$N = A\pi \left(\frac{\sin \Delta}{\sin d_1} - \frac{\sin \delta}{\tan d_1} \right)$$

and then

$$a = -A\pi \frac{\sin \delta}{\tan d_1} \qquad b = A\pi \frac{\sin \Delta}{\sin d_1}$$

we shall have

$$N = a + b \qquad (459)$$

and a and b can be taken from Table XIX. where a is called "the first part of N," and b "the second part of N." We then have

$$d - d_1 = N \sin \varphi \qquad (460)$$

which is the correction to be added to d_1 to obtain the geocentric distance d. Table XIX. is computed with the mean value of

$\pi = 57' \ 30''$, which will not produce more than $1''$ error in $d - d_1$ in any case. But, if we wish to compute the correction for the actual parallax, we shall have, after finding N by the table,

$$d - d_1 = N \sin \varphi \times \frac{\pi}{3450''} \qquad (460^*)$$

π being in seconds.

The trouble of finding the declinations of the bodies and the use of Table XIX. would be saved if the Almanac contained the logarithm of N in connection with the lunar Ephemeris. The value of $\log N$ in the Almanac would, of course, be computed with the actual parallax, and (460) would be perfectly exact.

We have yet to introduce corrections for the elliptical figure of the discs of the moon and sun produced by refraction. These corrections are obtained by Tables XVII. and XVIII., which are constructed upon the following principles. Let

$\Delta s_1, \Delta S_1 = $ the contractions of the vertical semidiameters,
$\Delta s, \Delta S = $ the contractions of the inclined semidiameters;

then we have (Art. 133)

$$\Delta s = \Delta S_1 \cos^2 q \qquad\qquad \Delta S = \Delta S_1 \cos^2 Q$$

where $q = $ the angle $ZM'S'$ (Fig. 29) and $Q = ZS'M'$. We have

$$\cos q = \frac{\sin H' - \sin h' \cos d'}{\cos h' \sin d'}$$

But, by (456),

$$\frac{\sin H'}{\cos h' \sin d'} = -\frac{B'}{B(\pi_1 - r') \cos h'} \qquad \frac{\sin h' \cos d'}{\cos h' \sin d'} = \frac{A'}{A(\pi_1 - r') \cos h'}$$

so that

$$\cos q = -\left(\frac{A'}{A} + \frac{B'}{B}\right) \frac{1}{(\pi_1 - r') \cos h'}$$

If we put $A = 1$ and $B = 1$, which are approximate values, we shall have

$$\cos q = -\frac{A' + B'}{(\pi_1 - r') \cos h'}$$

$$\Delta s = \Delta s_1 \left[\frac{A' + B'}{(\pi_1 - r') \cos h'}\right]^2 \qquad (461)$$

$$\Delta s = (A' + B')^2 \times \frac{g}{f}$$

For the sun, Table XVIII.A, with the arguments H' and $R' - P$, gives the value of

$$G = \frac{(R' - P)^2 \cos^2 H'}{\Delta S_1} \times F$$

in which $F = \dfrac{1}{200}$; and Table XVIII.B gives

$$\Delta S = (C' + D')^2 \times \frac{F}{G}$$

In these tables $A' + B'$ is called the "whole correction of the moon," and $C' + D'$ the "whole correction of the sun." As these quantities are furnished by the previous computation of the true distance, the required corrections are taken from the tables without any additional computation.

The values of Δs and ΔS are applied to the distance as follows: when the limb of the moon nearest to the star or planet is observed, Δs is to be subtracted, and when the farthest limb is observed, Δs is to be added; when the sun is observed, both Δs and ΔS are to be subtracted from d.

In strictness, these corrections should be applied to the distance d', and the distance thus corrected should be employed in computing the values of A', B', C', and D'. This would require a repetition of the computation after Δs and ΔS had been found by a first computation; but this repetition will rarely change the result by $0''.5$. In the extreme and improbable case when the distance is only $20°$ and one body is at the altitude $5°$ and the other directly above it in the same vertical circle (so that the entire contraction of the vertical semidiameter comes into account), such a repetition would change the result only $1''.8$; and even this error is much less than the probable error of sextant observations at this small altitude, where the sun and moon already cease to present perfectly defined discs.

250. I shall now recapitulate the steps of this method.

1st. The local mean time of the observation being T, and the assumed longitude L, take from the Ephemeris, for the approxi-

mate Greenwich time $T + L$, the quantities s, S, π, P, δ, and Δ. (For the sun we may always take $P = 8''.5$; for a star, $S = 0$, $P = 0$.)

2d. If h'', H'', d'' denote the observed altitudes and distance of the limbs, find

$$s' = s + \text{correction of Table XII.,}$$
$$\pi_1 = \pi + \text{correction of Table XIII.,}$$

and the apparent altitudes and distance of the centres,

$$h' = h'' \mp s', \qquad H' = H'' \mp S, \qquad d' = d'' \pm s' \pm S$$

upper signs for upper and nearest limbs, lower signs for lower and farthest limbs.

For the altitudes h' and H', take the "reduced refractions" r' and R' from Table XIV., correcting them by Table XIV.A and B for the barometer and thermometer. Then compute the quantities

$$A' = \quad (\pi_1 - r') A \sin h' \cot d' \qquad C' = -(R' - P) C \sin H' \cot d'$$
$$B' = - (\pi_1 - r') B \sin H' \operatorname{cosec} d' \qquad D' = \quad (R' - P) D \sin h' \operatorname{cosec} d'$$

for which the logarithms of A, B, C, and D are taken from Table XV. In this table the argument $\pi_1 - r'$ is called the "reduced parallax and refraction of the moon," and $R' - P$ the "reduced refraction and parallax of the sun (or planet) or star." For a star this argument is simply R'.

When $d' > 90°$, the signs of A' and C' will be reversed. It may be convenient for the computer to determine the signs by referring to the following table:

	A'	B'	C'	D'
$d' < 90°$	+	−	−	+
$d' > 90°$	−	−	+	+

3d. The terms A' and B', which depend upon the moon's parallax and refraction, may be called the first and second parts of the moon's correction, and the sum $A' + B'$ the "whole correction of the moon." In like manner, C' and D' may be called the first and second parts of the sun's, planet's, or star's correc-

tion, and the sum $C' + D'$ the "whole correction of the sun, planet, or star."

The sum of these corrections $= A' + B' + C' + D'$ may be called the "first correction of the distance." Taking it as the upper argument in Table XVI., find the second correction $= x$, the sign of which is indicated in the table.

4th. Take from Table XVII.A and B the contraction of its inclined semidiameter $= \Delta s$. If the sun is the other body, take also the contraction from Table XVIII.A and B, $= \Delta S$. The sign of either of these corrections will be positive when the farthest limb is observed, and negative when the nearest limb is observed.

5th. The correction for the compression of the earth is $= N \sin \varphi$, φ being the latitude; and N may be accurately computed by the formula

$$N := A\pi\left(\frac{\sin \Delta}{\sin d_1} - \frac{\sin \delta}{\tan d_1}\right)$$

or it may be found within $1''$ by Table XIX., the mode of consulting which is evident. The sign of $N \sin \varphi$ will be determined by the signs of N and $\sin \varphi$, remembering that for south latitudes $\sin \varphi$ is negative.

All the corrections being applied to d', we have the geocentric distance d; and hence the corresponding Greenwich time and the longitude.

EXAMPLE.—Let us take the example of the preceding article (p. 399), in which the observation gives

1856, March 9th, $\phi = 35°$.

T	$= 5^h 14^m 6^s$	$\mathcal{D}\, h'' = 52° 34' 0''$	Barom. 29.5 in.
Assumed L	$= 10\ \ 0\ \ 0$	$\odot H'' = 8\ \ 56\ 23$	Therm. 58° F.
Approx. Gr. T.	$= 15\ 14\ 6$	$\mathcal{D}\, \odot\, d'' = 44\ \ 36\ 58.6$	

By the Ephemeris, we have

$s = 16' 23''.1$	$\pi = 60' 1''.9$	$S = 16' 8''.0$	$P = 8''.6$
Table XII. $+ 14\ .0$	Tab. XIII. $+ 3\ .9$	$\delta = + 14°$	$\Delta = -4°$
$s' = 16\ 37\ .1$	$\pi_1 = 60\ 5\ .8$		

The computation may be arranged as follows:

251. In consequence of the neglect of the fractions of a second in several parts of the above method, it is possible that the computed distance may be in error several seconds, but it is easily seen that the error from this cause will be most sensible in cases where the distance is small; and, since the lunar distances are given in the Ephemeris for a number of objects, the observer can rarely be obliged to employ a small distance. If he confines himself to distances greater than 45° (as he may readily do), the method will rarely be in error so much as 2″, especially if he also avoids altitudes less than 10°. When we remember that the least count of the sextant reading is 10″, and that to the probable error of observation we must add the errors of gradua-tion, of eccentricity, and of the index correction, it must be con-ceded that we cannot hope to reduce the probable error of an observed distance below 5″, if indeed we can reduce it below 10″. Our approximate method is, therefore, for all practical purposes, a perfect method, in relation to our present means of observation.

252. If the altitudes have not been observed, they may be computed from the hour angles and declinations of the bodies, the hour angles being found from the local time and the right ascensions. But the declination and right ascension of the moon will be taken from the Ephemeris for the approximate Green-wich time found with the assumed longitude. If, then, the assumed longitude is greatly in error, a repetition of the computation may be necessary, starting from the Greenwich time furnished by the first. As a practical rule, we may be satisfied with the first computation when the error in the assumed longitude is not more than 30′. In the determination of the longitude of a fixed point on land, it will be advisable to omit the observation of the altitudes, as thereby the observer gains time to multiply the observations of the distance. But at sea, where an immediate result is required with the least expenditure of figures, the alti-tudes should be observed.

253. At sea, the observation is noted by a chronometer regu-lated to Greenwich time, and the most direct employment of the resulting Greenwich time will then be to determine the true correction of the chronometer. This proceeding has the advan-

by differentiating which we find

$$\delta d = \frac{\cos \delta \, \cos \Delta \, \sin (a - A)}{\sin d} \cdot \delta a$$

$$- \frac{\cos \delta \sin \Delta - \sin \delta \cos \Delta \cos (a - A)}{\sin d} \cdot \delta \delta \qquad (464)$$

If then

$$v = \text{the change of distance in } 3^h,$$

we shall have

$$\delta L = - \delta d \times \frac{3^h}{v} \qquad (465)$$

in computing which we employ the proportional logarithm of the Ephemeris, $Q = \log \dfrac{3^h}{v}$, reduced to the time of the observation.

EXAMPLE.—At the time of the observation computed in Art. 250, we have

Moon, $a =$	$2^h \, 11^m \, 14^s$		$\delta = + 14° \, 18'.4$	
Sun, $A =$	$23 \quad 22 \quad 25$		$\Delta = - \quad 4 \quad 3.1$	
$a - A =$	$2 \quad 49 \quad 19$		$d = \quad 45 \quad 48.2$	
$=$	$42° \, 19'.8$			

with which we find, by (464),

$$\delta d = 0.908 \, \delta a + 0.350 \, \delta \delta$$

and hence, by (465), with log $Q = 0.2511$,

$$\delta L = - 1.62 \, \delta a - 0.62 \, \delta \delta$$

Suppose then we find from the Greenwich observations $\delta a = - 0^s.38 = - 5''.7$ and $\delta \delta = - 4''.0$, the correction of the longitude above found will be

$$\delta L = + 11^s.7$$

255. *To find the longitude by a lunar distance not given in the Ephemeris.*—The regular lunar-distance stars mentioned in Art. 247 are selected nearly in the moon's path, and are therefore in general most favorable for the accurate determination of the Greenwich time. Nevertheless, it may occasionally be found expedient to employ other stars, not too far from the ecliptic. Sometimes, too, a different star may have been observed by mistake, and it may be important to make use of the observation.

The true distance d is to be found from the observed distance by the preceding methods, as in any other case. Let the local time of the observation be T, and the assumed longitude L. Take from the Ephemeris the moon's right ascension α and declination δ for the Greenwich time $T + L$, and also the star's right ascension A and declination $\varDelta$; with which the corresponding true distance d_0 is found by the formula

$$\cos d_0 = \sin \delta \sin \varDelta + \cos \delta \cos \varDelta \cos (\alpha - A)$$

Then, if $d = d_0$, the assumed longitude is correct; if otherwise, put

$\lambda =$ the increase of α in one minute of mean time,
$\beta =$ the increase of δ "   "   "   "
$\gamma =$ the increase of d "   "   "   "

then we have, by (464),

$$\gamma = \frac{\cos \delta \cos \varDelta \sin (\alpha - A)}{\sin d_0} \cdot \lambda - \frac{\cos \delta \sin \varDelta - \sin \delta \cos \varDelta \cos (\alpha - A)}{\sin d_0} \cdot \beta$$

and hence the correction of the assumed longitude in seconds of time,

$$\delta L = \frac{60}{\gamma} (d - d_0)$$

For computation by logarithms, these formulæ may be arranged as follows:

$$\left. \begin{aligned} & \tan M = \frac{\tan \varDelta}{\cos (\alpha - A)} \\[2mm] & \cos d_0 = \frac{\sin \varDelta \cos (\delta - M)}{\sin M} \\[2mm] & \gamma = \lambda \cdot \frac{\cos \delta \cos \varDelta \sin (\alpha - A)}{\sin d_0} + \beta \cdot \cot d_0 \tan (\delta - M) \\[2mm] & \delta L = \frac{60 (d - d_0)}{\gamma} \end{aligned} \right\} \quad (466)$$

EXAMPLE.—Suppose an observer has measured the distance of the moon from *Arcturus*, at the local mean time 1856 March 16, $T = 10^h 30^m 0^s$, in the assumed longitude $L = 6^h 0^m 0^s$, and, reducing his observation, finds the true distance

$$d = 73° 55' 10''$$

what is the true longitude?

the instant of the observation. The data from the Ephemeris required in computing the local time are taken for the Greenwich time given by the chronometer.

EXAMPLE.—A ship being about to sail from New York, the master determined the correction on Greenwich time and the rate of his chronometer by observations on two dates, as follows:

1860 April 22, at Greenwich noon, chron. correction $= + 3^m 10^s.0$
" " 30, " " " " $= + 3\ 43\ .6$
Rate in 8 days $=\quad + 33\ .6$
Daily rate $=\quad + 4\ .2$

On May 18 following, about $7^h 30^m$ A.M., the ship being in latitude 41° 33′ N., three altitudes of the sun's lower limb were observed from the sea horizon as below. The correction of the chronometer on that day is found from the correction on April 30 by adding the rate for 18 days. (It will not usually be worth while to regard the fraction of a day in computing the total rate at sea.) The record of the observation and the whole computation may be arranged as follows:

1860 May 18. $\phi = 41° 33′$

Chronometer	$9^h 37^m 21^s.$		☉ 29° 40′ 10″	Barom. 30.32^{in}.	
	" 37 53.		" 46 0	Therm. 59° F.	
	" 38 20.		" 50 50		
Mean	$= 9\ 37\ 51.8$		Mean $= 29\ 45\ 40$		
Correction	$= + 4\ 59.2$		Index corr. $= - 1\ 10$		
Gr. date $=$ May 17,	21 42 50.5		Dip $= - 4\ 2$		

for which time we take from the Ephemeris the quantities

☉'s $\delta = 19° 38′ 39″$
Semidiameter $= 15′ 50″$
Equation of time $= - 8^m 49^s.8$

29 40 28
Semid. $= + 15\ 50$
Refraction $= - 1\ 42$
Parallax $= + 8$
$h = 29\ 54\ 44$
$\phi = 41\ 33\ 0$ sec. 0.12588
$P = 70\ 21\ 21$ sec. 0.02604
$s = 70\ 54\ 33$ cos 9.51464
$s - h = 40\ 59\ 49$ cos 9.81692

9.48348

Apparent time $= 7^h 32^m 6^s.3$ sin 9.74174
Eq. of time $= - 3\ 49.8$
Local mean time $= 19\ 28\ 16.5$
Gr. " " $= 21\ 42\ 50.5$
Longitude $= 2\ 14\ 34 = 33° 38′.5$ W.

In this observation, the sun was near the prime vertical, a position most favorable to accuracy (Art. 149).

The method by equal altitudes may also be used for finding the time at sea in low latitudes, as in Arts. 158, 159.

258. In order that the longitude thus found shall be worthy of confidence, the greatest care must be bestowed upon the determination of the rate. As a single chronometer might deviate very greatly without being distrusted by the navigator, it is well to have at least three chronometers, and to take the mean of the longitudes which they severally give in every case.

But, whatever care may have been taken in determining the rate on shore, the sea rate will generally be found to differ from it more or less, as the instrument is affected by the motion of the ship; and, since a cause which accelerates or retards one chronometer may produce the same effect upon the others, the agreement of even three chronometers is not an absolutely certain proof of their correctness. The sea rate may be found by determining the chronometer correction at two ports whose *difference* of longitude is well known, although the absolute longitudes of both ports may be somewhat uncertain. For this purpose, a " Table of Chronometric Differences of Longitude" is given in RAPER'S *Practice of Navigation*, the use of which is illustrated in the following example.

EXAMPLE.—At St. Helena, May 2, the correction of a chronometer on the local time was $+ 0^h 23^m 10^s.3$. At the Cape of Good Hope, May 17, the correction on the local time was $- 1^h 14^m 28^s.6$; what was the sea rate?
We have

Corr. at St. Helena, May 2d	$= + 0^h 23^m 10^s.3$	
Chron. diff. of long. from Raper	$= - 1 \quad 36 \quad 45$.	
Corr. for Cape of G. H., May 2d	$= - 1 \quad 13 \quad 34.7$	
" " " " 17th	$= - 1 \quad 14 \quad 28.6$	
Rate in 15 days $=$	$- \quad 53.9$	
Daily sea rate $=$	$- \quad 3.59$	

259. *By lunar distances.*—Chronometers, however perfectly made, are liable to derangement, and cannot be implicitly relied upon in a long voyage. The method of lunar distances (Arts. 247–256) is, therefore, employed as an occasional check upon the chronometers even where the latter are used for finding the longitude from day to day. When there is no chronometer on

board, the method of lunar distances is the only regularly available method for finding the longitude at sea, at once sufficiently accurate and sufficiently simple.

As a check upon the chronometer, the result of a lunar distance is used as in Art. 253.

In long voyages an assiduous observer may determine the sea rates of his chronometers with considerable precision. For this purpose, it is expedient to combine observations taken at various times during a lunation in such a manner as to eliminate as far as possible constant errors of the sextant and of the observer (Art. 256). Suppose distances of the sun are employed exclusively. Let two chronometer corrections be found from two nearly equal distances measured on opposite sides of the sun on two different dates, in the first and second half of the lunation respectively. The mean of these corrections will be the correction for the mean date, very nearly free from constant instrumental and personal errors. In like manner, any number of *pairs* of equal, or nearly equal, distances may be combined, and a mean chronometer correction determined for a mean date from all the observations of the lunation. The sea rate will be found by comparing two corrections thus determined in two different lunations. This method has been successfully applied in voyages between England and India.

260. *By the eclipses of Jupiter's satellites.*—An observed eclipse of one of Jupiter's satellites furnishes immediately the Greenwich time without any computation (Art. 225.) But the eclipse is not sufficiently instantaneous to give great accuracy; for, with the ordinary spy-glass with which the eclipse may be observed on board ship, the time of the disappearance of the satellite may precede the true time of total eclipse by even a whole minute. The time of disappearance will also vary with the clearness of the atmosphere. Since, however, the same causes which accelerate the disappearance will retard the reappearance, if both phenomena are observed on the same evening under nearly the same atmospheric conditions, the mean of the two resulting longitudes will be nearly correct. Still, the method has not the advantage possessed by lunar distances of being almost always available at times suited to the convenience of the navigator.

261. *By the moon's altitude.*—This method, as given in Art. 243,

is vertical to an observer at the point M of the globe (Fig 32).
Let a small circle $AA'A''$ be described on
the globe from M as a pole, with a polar dis-
tance MA equal to the zenith distance, or
complement of the observed altitude, of the
sun. It is evident that at all places *within*
this circle an observer would at the given
time observe a smaller zenith distance, and
at all places without this circle a greater
zenith distance; and therefore the observa-
tion fully determines the observer to be *on*
the circumference of the small circle $AA'A''$. If, then, the
navigator can project this small circle upon an artificial globe or
a chart, *the knowledge that he is upon this circle will be just as valuable
to him in enabling him to avoid dangers as the knowledge of either his
latitude alone or his longitude alone;* since one of the latter elements
only determines a point to be in a certain circle, without fixing
upon any particular point of that circle.

Fig. 32.
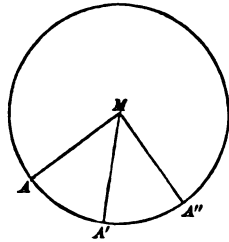
The small circle of the globe described from the projection of
the celestial object as a pole we shall call a *circle of position*.

264. *To find the place on the globe at which the sun is vertical (or the
sun's projection on the globe) at a given Greenwich time.*—The sun's
hour angle from the Greenwich meridian is the Greenwich
apparent time. The diurnal motion of the earth brings the sun
into the zenith of all the places whose latitude is just equal to
the sun's declination. Hence the required projection of the
sun is a place whose longitude (reckoned westward from Green-
wich from 0^h to 24^h) is equal to the Greenwich apparent time,
and whose latitude is equal to the sun's declination at that time.

265. *From an altitude of the sun taken at a given Greenwich time,
to find the circle of position of the observer, by projection on an artificial
globe.*—Find the Greenwich apparent time and the sun's declina-
tion, and put down on the globe the sun's projection by the
preceding article. From this point as a pole, describe a small
circle with a circular radius equal to the true zenith distance
deduced from the observation. This will be the required circle
of position.

266. The preceding problem may be extended to any celestial

object. The pole of the circle of position will always be the place whose west longitude is the Greenwich hour angle of the object (reckoned from 0^h to 24^h) and whose latitude is the declination of the object. The hour angle is found by Art. 54.

267. *To find both the latitude and the longitude of a ship by circles of position projected on an artificial globe.*—*First.* Take the altitudes of two different objects at the same time by the Greenwich chronometer. Put down on the globe, by the preceding problem, their two circles of position. The observer, being in the circumference of each of these circles, must be at one of their two points of intersection; which of the two, he can generally determine from an approximate knowledge of his position.

Second. Let the same object be observed at two different times, and project a circle of position for each. Their intersection gives the position of the ship as before. If between the observations the ship has moved, the first altitude must be reduced to the second place of observation by applying the correction of Art. 209, formula (380). The projection then gives the ship's position at the second observation.

268. *From an altitude of a celestial body taken at a given Greenwich time, to find the circle of position of the observer, by projection on a Mercator chart.*—The scale upon which the largest artificial globes are constructed is much smaller than that of the working charts used by navigators. But on the Mercator chart a circle of
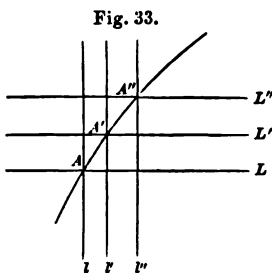
Fig. 33.

position will be distorted, and can only be laid down by points. Let L, L', L'' (Fig. 33) be any parallels of latitude crossed by the required circle. For each of these latitudes, with the true altitude found from the observation and the polar distance of the celestial body taken for the Greenwich time, compute the local time, and hence the longitude, "by chronometer" (Art. 257). Let l, l', l'' be the longitudes thus found. Let A, A', A'' be the points whose latitudes and longitudes are, respectively, L, l; L', l'; L'', l''; these are evidently points of the required circle. The ship is consequently in the curve $AA'A''$, traced through these points.

so great as to affect sensibly the declination which is taken from the Ephemeris for the time given by the chronometer. This method is, therefore, a convenient substitute for the usual method of finding the latitude at sea by two altitudes, a projection on the sailing chart being always sufficient for the purposes of the navigator.

Instead of reducing the first altitude for the change of the ship's position between the observations, we may put down the circle of position for each observation and afterwards shift one of them by a quantity due to the ship's run. Thus, let the first observation give the position line AA' (Fig. 35), and let Aa represent, in direction and length, the ship's course and distance sailed between the observations. Draw aa' parallel to AA'. Then, BB' being the position line by the second observation, its intersection C with aa' is the required position of the ship at the second observation.

Fig. 35.
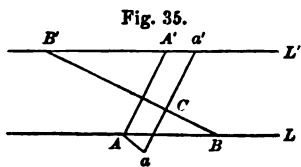
270. If the latitude is desired by computation, independently of the projection, it is readily found as follows. Let

l_1, l_2 = the longitudes (of A and B) found from the first and second altitudes respectively with the latitude L,

l_1', l_2' = the longitudes (of A' and B') found from the same altitudes with the latitude L',

L_0 = the latitude of C.

From Fig. 34 we have, by the similarity of the triangles ABC and $A'B'C$,

$$l_2' - l_1' : l_1 - l_2 = B'C : BC$$

whence

$$(l_2' - l_1') + (l_1 - l_2) : l_1 - l_2 = BB' : BC = L' - L : L_0 - L$$

$$L_0 = L + \frac{(L' - L)(l_1 - l_2)}{(l_2' - l_1') + (l_1 - l_2)} \qquad (467)$$

This formula reduces SUMNER's method of "double altitudes" to that given long ago by LALANDE (*Astronomie*, Art. 3992, and *Abrégé de Navigation*, p. 68). The distinctive feature of SUMNER's process, however, is that a single altitude taken at any time is made available for determining a line of the globe on which the ship is situated.

more concentric circles on the ground from the foot of the stake as a centre. At the two instants before and after noon when the shadow of the stake extends to the same circle, the azimuths of the shadow east and west are equal. The points of the circle at which the shadow terminates at these instants being marked, let the included arc be bisected; the point of bisection and the centre of the stake then determine the meridian line. Theoretically, a small correction should be made for the sun's change of declination, but it would be quite superfluous in this method.

275. *By single altitudes.*—With an altitude and azimuth instrument, observe the altitude of a star at the instant of its passage over the middle vertical thread (at any time), and read the horizontal circle. Correct the observed altitude for refraction. Then, if

h = the true altitude,
φ = the latitude of the place of observation,
p = the star's polar distance,
A = the star's azimuth,
A' = the reading of the horizontal circle,

we have, from the triangle formed by the zenith, the pole, and the star,

$$\tan^2 \tfrac{1}{2} A = \frac{\sin (s - \varphi) \sin (s - h)}{\cos s \cos (s - p)} \tag{468}$$

in which

$$s = \tfrac{1}{2}(\varphi + h + p)$$

In this formula the latitude may be taken with the positive sign, whether nórth or south, and p is then to be reckoned from the elevated pole; consequently, also, A will be the azimuth reckoned from the elevated pole.

It is evident that in order to bring the telescope into the plane of the meridian we have only to revolve the instrument through the angle A, and therefore either $A' + A$ or $A' - A$, according to the direction of the graduations of the circle, will be the reading of the horizontal circle when the telescope is in the meridian.

The same method can be followed when the azimuth is observed with a compass and the altitude is measured with a sextant; and then $A' - A$ is the *variation of the compass.*

276. From the first equation of (50), φ and δ being constant, we have

$$dA = -\frac{dh}{\cos h \tan q}$$

and therefore an error in the observed altitude will have the least effect upon the computed azimuth when tan q is a maximum; that is, when the star is on the prime vertical. Therefore, in the practice of the preceding method the star should be as far from the meridian as possible.

277. *By equal altitudes of a star.*—Observe the azimuth of a star with an altitude and azimuth instrument, or a compass, when at the same altitude east and west of the meridian. The mean of the two readings of the instrument is the reading when its sight line is in the direction of the meridian. This is the method of Article 274, rendered accurate by the introduction of proper instruments for observing both the altitude and the azimuth.

278. If equal altitudes of the sun are employed, a correction for the change of the sun's declination is necessary, since equal azimuths will no longer correspond to equal altitudes. Let

A' = the east azimuth at the first observation,
A = " west " " second "
δ = the declination at noon,
$\Delta\delta$ = the increase of declination from the first to the second observation,

then, by (1), we have, h being the altitude in each case,

$$\sin (\delta - \tfrac{1}{2} \Delta\delta) = \sin \varphi \sin h - \cos \varphi \cos h \cos A'$$
$$\sin (\delta + \tfrac{1}{2} \Delta\delta) = \sin \varphi \sin h - \cos \varphi \cos h \cos A$$

the difference of which gives

$$2 \cos \delta \sin \tfrac{1}{2} \Delta\delta = 2 \cos \varphi \cos h \sin \tfrac{1}{2} (A + A') \sin \tfrac{1}{2} (A - A')$$

whence, since $\Delta\delta$ is but a few minutes, we have, with sufficient accuracy,

$$A - A' = \frac{\Delta\delta \cos \delta}{\cos \varphi \cos h \sin A} \tag{469}$$

It will be necessary to note the times of the two observations in order to find $\Delta\delta$. ·If we take half the elapsed time as the hour angle t of the western observation, we shall have, instead of (469), the more convenient formula

$$A - A' = \frac{\Delta\delta}{\cos\varphi\sin t} \qquad (470)$$

It will not be necessary to know the exact value of h, if only the same *instrumental* altitude is employed at both observations.

Now let A_1' and A_1 be the readings of the horizontal circle at the two observations, then the readings corresponding to equal azimuths are

$$A_1' \text{ and } A_1 - (A - A')$$

and, consequently, the reading for the meridian is the mean of these, or

$$\tfrac{1}{2}(A_1' + A_1) - \tfrac{1}{2}(A - A')$$

That is, the reading for the meridian is the mean of the observed readings diminished by one-half the correction (470). We here suppose the graduations to proceed from 0° to 360°, and from left to right.

279. *By the angular distance of the sun from any terrestrial object.*— If the true azimuth of any object in view is known, the direction of the meridian is, of course, known also. The following method can be carried out with the sextant alone. Measure the angular distance of the sun's limb from any well-defined point of a distant terrestrial object, and note the time by a chronometer. Measure also the angular height of the terrestrial point above the horizontal plane. The correction of the chronometer being known, deduce the local apparent time, or the sun's hour angle t (Art. 54), and then with the sun's declination δ and the latitude φ compute the true altitude h and azimuth A of the sun by the formulæ (16), or

$$\tan M = \frac{\tan\delta}{\cos t}, \quad \tan A = \frac{\tan t \cos M}{\sin(\varphi - M)}, \quad \tan h = \cot(\varphi - M)\cos A \quad (471)$$

Now, let O, Fig. 37, be the apparent position of the terrestrial point, projected upon the celestial sphere; S the apparent place of the sun, Z the zenith, P the pole; and put

D = the apparent angular distance of the sun's centre from the terrestrial point
= the observed distance increased by the sun's semidiameter,
H = the apparent altitude of the point,
h' = the sun's apparent altitude,
a = the difference of the azimuth of the sun and the point,
A' = the azimuth of the point.

Fig. 37.
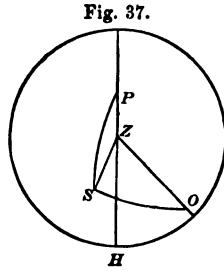
The apparent altitude h' will be deduced from the true altitude by adding the refraction and subtracting the parallax. Then in the triangle SZO we have given the three sides $ZS = 90° - h'$, $ZO = 90° - H$, $SO = D$, and hence the angle $SZO = a$ can be found by the formula

$$\tan^2 \tfrac{1}{2} a = \frac{\sin(s - H)\sin(s - h')}{\cos s \cos(s - D)} \qquad (472)$$

in which

$$s = \tfrac{1}{2}(H + h' + D)$$

Then we have

$$A' = A \pm a \qquad (473)$$

and the proper sign of a to be used in this equation must be determined by the position of the sun with respect to the object at the time of the observation.

If the altitude of the sun is observed, we can dispense with the computation of (471), and compute A by the formula (468). The chronometer will not then be required, but an approximate knowledge of the local time and the longitude is necessary in order to find δ from the Ephemeris.

If the terrestrial object is very remote, it will often suffice to regard its altitude as zero, and then we shall find that (472) reduces to

$$\tan \tfrac{1}{2} a = \sqrt{[\tan \tfrac{1}{2}(D + h') \tan \tfrac{1}{2}(D - h')]} \qquad (474)$$

This method is frequently used in hydrographic surveying to determine the meridian line of the chart.

EXAMPLE.—From a certain point B in a survey the azimuth of a point C is required from the following observation:

Chronometer time	= $4^h\ 12^m\ 12^s$	
Chronom. correction	= $-\ 2\quad 0$	
Local mean time	= $4\quad 10\quad 12$	
Equation of time	= $-\ 4\quad 10.9$	
Local app. time, t	= $4\quad 5\quad 1.1$	

Altitude of $C = H =$ 0° 30′ 20″
Distance of the nearest limb of the sun from the point $C =$ 48° 17′ 10″
Semidiameter = 16 1
$D =$ 48 33 11

in which A is the azimuth reckoned from the elevated pole. At this instant the star's azimuth reaches its maximum, and for a certain small interval of time appears to be stationary, so that the observer has time to set his instrument accurately upon the star.

In order to be prepared for the observation, the time of the elongation must be (at least approximately) known. The hour angle of the star is found by the formula

$$\cos t = \frac{\tan \varphi}{\tan \delta}$$

and from t and the star's right ascension the local time is found, Art. 55.

The pole star is preferred, on account of its extremely slow motion.

If the latitude is unknown, the direction of the meridian may nevertheless be obtained by observing the star at both its eastern and its western greatest elongations. The mean of the readings of the horizontal circle at the two observations is the reading for the meridian.

283. One of the most refined methods of determining the direction of the meridian is that by which the transit instrument is adjusted, or by which its deviation from the plane of the meridian is measured; for which see Vol. II.

284. At sea, the direction of the meridian, or the variation of the compass, is found with sufficient accuracy by the graphic process of Art. 271.

NS. Let S', M', be the centres of the sun and moon when at their least true distance, and put

> $\beta =$ the moon's latitude at conjunction $= SM$,
> $I =$ the inclination of the moon's orbit to the ecliptic,
> $\lambda =$ the quotient of the moon's motion in longitude divided by the sun's,
> $\Sigma =$ the least true distance $= S'M'$,
> $\gamma =$ the angle SMS'.

We may regard NMS as a plane triangle; and, drawing $M'P$ perpendicular to NS, we find

$$SS' = \beta \tan \gamma \qquad SP = \lambda \beta \tan \gamma$$

and hence

$$S'P = \beta (\lambda - 1) \tan \gamma \qquad M'P = \beta - \lambda \beta \tan \gamma \tan I$$
$$\Sigma^2 = \beta^2 [(\lambda - 1)^2 \tan^2 \gamma + (1 - \lambda \tan I \tan \gamma)^2]$$

To find the value of γ for which this expression becomes a minimum, we put its derivative taken relatively to γ equal to zero, whence

$$\tan \gamma = \frac{\lambda \tan I}{(\lambda - 1)^2 + \lambda^2 \tan^2 I}$$

which substituted in the value of Σ^2 reduces it to

$$\Sigma^2 = \frac{\beta^2 (\lambda - 1)^2}{(\lambda - 1)^2 + \lambda^2 \tan^2 I}$$

If then we assume I' such that

$$\tan I' = \frac{\lambda}{\lambda - 1} \tan I \qquad\qquad (475)$$

we have for the least true distance

$$\Sigma = \beta \cos I' \qquad\qquad (476)$$

The apparent distance of the centres of the sun and moon as seen from the surface of the earth may be less than Σ by the difference of the horizontal parallaxes of the two bodies: so that if we put

> $\pi =$ the moon's horizontal parallax,
> $\pi' =$ the sun's   "   "

we have

$$\text{minimum apparent distance} = \Sigma - (\pi - \pi')$$

An eclipse will occur when this least apparent distance of the centres is less than the sum of the semidiameters of the bodies; and therefore, putting

$s = $ the moon's semidiameter,
$s' = $ the sun's   "

we shall have, in case of eclipse,

$$\Sigma - (\pi - \pi') < s + s'$$

or

$$\beta \cos I' < \pi - \pi' + s + s' \tag{477}$$

This formula gives the required limit with great precision; but, since I' is small, its cosine does not vary much for different eclipses, and we may in most cases employ its mean value. We have, by observation,

	Greatest values.	Least values.	Mean values.
I	5° 20′ 6″	4° 57′ 22″	5° 8′ 44″
π	61′ 32″	52′ 50″	57′ 11″
π'	9	8	8.5
s	16 46	14 24	15 35
s'	16 18	15 45	16 1
λ	16.19	10.89	13.5

From the mean values of I and λ we find the mean value of $\sec I' = 1.00472$, and the condition (477) becomes

$$\beta < (\pi - \pi' + s + s') \times 1.00472$$

or

$$\beta < \pi - \pi' + s + s' + (\pi - \pi' + s + s') \times .00472$$

where the small fractional term varies between 20″ and 30″. Taking its mean value, we have, with sufficient precision for all but very unusual cases,

$$\beta < \pi - \pi' + s + s' + 25'' \tag{478}$$

If in this formula we substitute the greatest values of π, s, and s', and the least value of π', the limit

$$\beta < 1° \, 34' \, 53''$$

is the greatest limit of the moon's latitude at the time of conjunction, for which an eclipse can occur.

If in (478) we substitute the least values of π, s, and s', and the greatest value of π', the limit

$$\beta < 1° \, 23' \, 15''$$

is the least limit of the moon's latitude at the time of conjunction for which an eclipse can fail to occur.

Hence a solar eclipse is *certain* if at new moon $\beta < 1° \, 23' \, 15''$, *impossible* if $\beta > 1° \, 34' \, 53''$, and *doubtful* between these limits. For the doubtful cases we must apply (478), or for greater precision (477), using the actual values of π, π', s, s', λ, and I for the date.

EXAMPLE.—On July 18, 1860, the conjunction of the moon and sun in longitude occurs at $2^h \, 19^m.2$ Greenwich mean time: will an eclipse occur? We find at this time, from the Ephemeris,

$$\beta = 0° \, 33' \, 18''.6$$

which, being within the limit $1° \, 23' \, 15''$, renders an eclipse certain at this time.

Having thus found that an eclipse will be visible in some part of the earth, we can proceed to the exact computation of the phenomenon. The method here adopted is a modified form of BESSEL'S,[*] which is at once rigorous in theory and simple in practice. For the sake of clearness, I shall develop it in a series of problems.

Fundamental Equations of the Theory of Eclipses.

288. *To investigate the condition of the beginning or ending of a solar eclipse at a given place on the earth's surface.*—The observer sees the limbs of the sun and moon in apparent contact when he is situated in the surface of a cone which envelops and is in contact with the two bodies. We may have two such cones:

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[*] See *Astronomische Nachrichten*, Nos. 151, 152, and, for the full development of the method with the utmost rigor, BESSEL'S *Astronomische Untersuchungen*, Vol. II. HANSEN'S development, based upon the same fundamental equations, but theoretically less accurate, may also be consulted with advantage: it is given in *Astronom. Nach.*, Nos. 839–342.

where the second members, besides the right ascensions and declinations, involve only the quantity b, which may be expressed in terms of the parallaxes as follows:

Let

$\pi =$ the moon's equatorial horizontal parallax,
$\pi' =$ the sun's   "   "   "

then we have (Art. 89)

$$b = \frac{r}{r'} = \frac{\sin \pi'}{\sin \pi}$$

If, further,

$\pi_0 =$ the sun's mean horizontal parallax,

and r' is expressed in terms of the sun's mean distance from the earth, we have, as in (146),

$$\sin \pi' = \frac{\sin \pi_0}{r'}$$

and hence

$$b = \frac{\sin \pi_0}{r' \sin \pi} \qquad (480)$$

which is the most convenient form for computing b, because r' and π are given in the Ephemeris, and π_0 is a constant.

290. The equations (479) are rigorously exact, but as b is only about $\frac{1}{400}$, and $a - a'$ at the time of an eclipse cannot exceed $1° 43'$, $a - a'$ is a small arc never exceeding $17''$, which may be found by a brief approximative process with great precision. The quotient of the first equation divided by the second gives

$$\tan (a - a') = - \frac{b \cos \delta \sec \delta' \sin (a - a')}{1 - b \cos \delta \sec \delta' \cos (a - a')}$$

where the denominator differs from unity by the small quantity $b \cos \delta \sec \delta' \cos (a - a')$; and, since δ and δ' are nearly equal, this small difference may be put equal to b, and we may then write the formula thus:*

$$a - a' = - \frac{b}{1 - b} \cos \delta \sec \delta' (a - a')$$

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* Developing the formula for $\tan (a - a')$ in series, we have

$$a - a' = - \frac{b \cos \delta \sec \delta' \sin (a - a')}{\sin 1''} - \frac{b^2 \cos^2 \delta \sec^2 \delta' \sin 2 (a - a')}{2 \sin 1''} - \&c.$$

where the second term cannot exceed $0''.04$, and the third term is altogether inap-

If we take $\cos(a - a') = 1$ and $\cos(\alpha - \alpha') = 1$, we have, from the second and third of (479),

$$g \cos d = \cos \delta' - b \cos \delta$$
$$g \sin d = \sin \delta' - b \sin \delta$$

whence

$$g \sin(d - \delta') = -b \sin(\delta - \delta')$$
$$g \cos(d - \delta') = 1 - b \cos(\delta - \delta')$$

from which follows

$$\tan(d - \delta') = -\frac{b \sin(\delta - \delta')}{1 - b \cos(\delta - \delta')}$$

or, nearly,*

$$d - \delta' = -\frac{b}{1-b}(\delta - \delta')$$

From the above we also have, with sufficient precision for the subsequent application of g, the formula

$$g = 1 - b$$

The formulæ which determine the point Z, together with the quantity G, will, therefore, be

$$
\left.
\begin{aligned}
a &= a' - \frac{b}{1-b} \cos \delta \sec \delta' (\alpha - \alpha') \\[2ex]
d &= \delta' - \frac{b}{1-b}(\delta - \delta') \\[2ex]
g &= 1 - b, \qquad\qquad G = r'g
\end{aligned}
\right\} \quad (481)
$$

and in many cases it will suffice to take the extremely simple forms

$$a = a' - b(\alpha - \alpha') \qquad d = \delta' - b(\delta - \delta')$$

291. *To find the distance of a given place of observation from the axis of the shadow at a given time.*—Let the positions of the sun,

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preciable. The formula adopted in the text is the same as

$$a - a' = -b \cos \delta \sec \delta'(\alpha - \alpha')(1 - b)^{-1}$$
$$= -b \cos \delta \sec \delta'(\alpha - \alpha') - b^2 \cos \delta \sec \delta'(\alpha - \alpha') - \&c.$$

which, since $\cos \delta \sec \delta'$ may in the second term be put equal to unity, differs from the complete series only by terms of the third order. The error of the approximate formula is, therefore, something less than $0''.01$.

* The error of this formula, as can be easily shown, will never exceed $0''.088$.

we have

$$\mu - a = \mu_1 - \omega$$

To find μ_1 we have only to convert the Greenwich mean time T into sidereal time and to subtract a.

By means of the formulæ (482) and (483) the co-ordinates of the moon and of the place of observation can be accurately computed for any given time. Now, the co-ordinates x and y of the moon are also those of every point of the axis of the shadow: so that if we put

$\Delta =$ the distance of the place of observation from the axis of the shadow,

we have, evidently,

$$\Delta^2 = (x - \xi)^2 + (y - \eta)^2 \tag{484}$$

[The co-ordinates z and ζ have also been found, as they will be required hereafter.]

292. The distance Δ may be determined under another form, which we shall hereafter find useful. Let M', Fig. 42, be the apparent position of the moon's centre in the celestial sphere as seen from the place of observation; P the north pole; Z the point where the axis of the cone of shadow meets the sphere, as in Fig. 41; M_1, C_1, the projections of the moon's centre and of the place of observation on the principal plane. The distance C_1M_1 is equal to Δ, and is the projection of the line joining the place of observation and the moon's centre. The plane by which this line is projected contains the axis of the cone of shadow, and its intersection with the celestial sphere is, therefore, a great circle which passes through Z, and of which ZM' is a portion. Hence it follows that C_1M_1 makes the same angle with the axis of y that $M'Z$ makes with PZ: so that if we draw C_1N and M_1N parallel to the axes of y and x respectively, and put
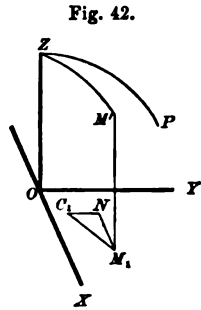
$$Q = PZM' = NC_1M_1$$

we have, from the right triangle C_1M_1N,

$$\left.\begin{array}{l} \Delta \sin Q = x - \xi \\ \Delta \cos Q = y - \eta \end{array}\right\} \tag{485}$$

the sum of the squares of which gives again the formula (484).

293. *To find the radius of the shadow on the principal plane, or on any given plane parallel to the principal plane.*—This radius is evidently equal to the distance of the vertex of the cone of shadow from the given plane, multiplied by the tangent of the angle of the cone. In Figs. 39 and 40, p. 440, let EF be the radius of the shadow on the principal plane, CD the radius on a parallel plane drawn through C. Let

$H =$ the apparent semidiameter of the sun at its mean distance,

$k =$ the ratio of the moon's radius to the earth's equatorial radius,

$f =$ the angle of the cone $= EVF$,

$c =$ the distance of the vertex of the cone above the principal plane $= VF$,

$\zeta =$ the distance of the given parallel plane above the principal plane $= DF$,

$l =$ the radius of the shadow on the principal plane $= EF$,

$L =$ the radius of the shadow on the parallel plane $= CD$.

If the mean distance of the sun from the earth is taken as unity, we have

the earth's radius $= \sin \pi_0$,

the moon's radius $= k \sin \pi_0 = MB$,

the sun's radius $= \sin H = SA$,

and, remembering that $G = r'g$ found by (481) is the distance MS, we easily deduce from the figures

$$\sin f = \frac{\sin H \pm k \sin \pi_0}{r'g} \qquad (486)$$

in which the upper sign corresponds to the penumbral and the lower to the umbral cone.

The numerator of this expression involves only constant quantities. According to BESSEL, $H = 959''.788$; ENCKE found $\pi_0 = 8''.57116$; and the value of k, found by BURCKHARDT from eclipses and occultations, is $k = 0.27227$;* whence we have

log $[\sin H + k \sin \pi_0] = 7.6688033$ for exterior contacts,
log $[\sin H - k \sin \pi_0] = 7.6666913$ for interior contacts.

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* The value of k here adopted is precisely that which the more recent investigation of OUDEMANS (*Astron. Nach.*, Vol. LI. p. 30) gives for eclipses of the sun. For occultations, a slightly increased value seems to be required.

may be taken as the conditions which determine the beginning or ending of an eclipse at a given place.

The equation (490), which is only expressed in a different form by (491), is to be regarded as the fundamental equation of the theory of eclipses.

295. By Art. 292, so long as $\varDelta$ is regarded as a positive quantity, Q is the position angle of the moon's centre at the point Z; and since the arc joining the point Z and the centre of the moon also passes through the centre of the sun, Q is the common position angle of both bodies.

Again, since in the case of a contact of the limbs the arc joining the centres passes through the point of contact, Q will also be the position angle of this point when all three points—sun's centre, moon's centre, and point of contact—lie on the same side of Z. In the case of total eclipse, however, the point of contact and the moon's centre evidently lie on opposite sides of the point Z; and if $l - i\zeta$ in (490) were a positive quantity, the angle Q which would satisfy these equations would still be the position angle of the moon's centre, but would differ 180° from the position angle of the point of contact. But, since we shall preserve the negative sign of $l - i\zeta$ for total eclipse (Art. 293), (and thereby give Q values which differ 180° from those which follow from a positive value), *the angle Q will in all cases be the position angle of the point of contact.*

296. The quantities a, d, x, y, l, and i may be computed by the formulæ (480), (481), (482), (486), (487), (488), for any given time at the first meridian, since they are all independent of the place of observation. In order to facilitate the application of the equations (490) and (491), it is therefore expedient to compute these general quantities for several equidistant instants preceding and following the time of conjunction of the sun and moon, and to arrange them in tables from which their values for any time may be readily found by interpolation.

The quantities x and y do not vary uniformly; and in order to obtain their values with accuracy from the tables for any time, we should employ the second and even the third differences in the interpolation. This is effected in the most simple manner by the following process. Let the times for which x and y have been computed be denoted by $T_0 - 2^h$, $T_0 - 1^h$, T_0, $T_0 + 1^h$,

$T_0 + 2^h$, the interval being one hour of mean time; and let the values of x and y for these times be denoted by x_{-2}, x_{-1}, &c., y_{-2}, y_{-1}, &c. Let the *mean* hourly changes of x and y from the epoch T_0 to any time $T = T_0 + \tau$ be denoted by x' and y'. Then the values of x' and y' for the instants $T_0 - 2^h$, $T_0 - 1^h$, &c. will be formed as in the following scheme, where c denotes the third difference of the values of x as found from the series x_{-2}, x_{-1}, &c. according to the form in Art. 69, and the difference for the instant T_0 is found by the first formula of (77). The form for computing y' is the same.

Time.	x	x'
$T_0 - 2^h$	x_{-2}	$\frac{1}{2}(x_0 - x_{-2})$
$T_0 - 1^h$	x_{-1}	$x_0 - x_{-1}$
T_0	x_0	$\frac{1}{2}(x_1 - x_{-1}) - \frac{1}{6}c$
$T_0 + 1^h$	x_1	$x_1 - x_0$
$T_0 + 2^h$	x_2	$\frac{1}{2}(x_2 - x_0)$

If then we require x and y for a time $T = T_0 + \tau$, we take x' and y' from the table for this time, and we have

$$x = x_0 + x'\tau$$
$$y = y_0 + y'\tau$$

297. EXAMPLE.—Compute the elements of the solar eclipse of July 18, 1860.

The mean Greenwich time of conjunction of the sun and moon in right ascension is July 18, $2^h 8^m 56^s$. The computation of the elements will therefore be made for the Greenwich hours 0, 1, 2, 3, 4, and 5. For these hours we take the following quantities from the American Ephemeris:

For the Moon.

Greenwich mean time.	a	δ	π
July 18, 0^h	116° 44′ 24″.30	21° 52′ 20″.3	59′ 45″.80
1	117 21 59 .10	42 32 .8	47 .13
2	117 59 30 .45	32 36 .4	48 .44
3	118 36 58 .35	22 31 .2	49 .72
4	119 14 22 .65	12 17 .2	50 .98
5	119 51 43 .35	1 54 .6	52 .22

For the Sun.

Greenwich mean time.	a'	δ'	log r'
July 18, 0^h	117° 59′ 41″.85	20° 57′ 56″.20	0.0069675
1	118 2 12 .50	57 29 .42	61
2	118 4 43 .14	57 2 .60	47
3	118 7 13 .77	56 35 .75	33
4	118 9 44 .39	56 8 .86	19
5	118 12 15 .00	55 41 .94	05

The formulæ to be employed will be here recapitulated, for convenient reference.

I. For the elements of *the point Z*:

$$b = \frac{\sin \pi_0}{r' \sin \pi} \qquad\qquad \log \sin \pi_0 = 5.61894$$

$$a = a' - \frac{b}{1-b} \cos \delta \sec \delta' (a - a') \quad \text{or, nearly,} \quad a = a' - b(a - a')$$

$$d = \delta' - \frac{b}{1-b} (\delta - \delta') \qquad\qquad \text{"} \qquad d = \delta' - b(\delta - \delta')$$

$$g = 1 - b$$

II. The moon's co-ordinates:

$$r = \frac{1}{\sin \pi}$$

$$x = r \cos \delta \sin (a - a)$$
$$y = r \sin (\delta - d) \cos^2 \tfrac{1}{2}(a - a) + r \sin (\delta + d) \sin^2 \tfrac{1}{2}(a - a)$$
$$z = r \cos (\delta - d) \cos^2 \tfrac{1}{2}(a - a) - r \cos (\delta + d) \sin^2 \tfrac{1}{2}(a - a)$$

III. The angle of the cone of shadow and the radius of the shadow:

For penumbra: or exterior contacts.	For umbra: or interior contacts
$\sin f = \dfrac{[7.668803]}{r'g}$	$\sin f = \dfrac{[7.666691]}{r'g}$

$$c = z + \frac{k}{\sin f}, \quad \log k = 9.435000, \quad c = z - \frac{k}{\sin f}$$

$$i = \tan f \qquad\qquad\qquad\qquad i = \tan f$$

$$l = ic \qquad\qquad\qquad\qquad\qquad l = ic$$

IV. The values of a, d, x, y, $\log i$, and l, will then be tabulated and the differences x' and y' formed according to Art. 296.

I give the computation for the three hours 1^h, 2^h, and 3^h, *in extenso.*

I. Elements of *the point Z*.

		1^h	2^h	3^h
	$a - a'$	$-0°\ 40'\ 13''.40$	$-0°\ 5'\ 12''.69$	$+0°\ 29'\ 44''.58$
	$\delta - \delta'$	$+\ 45\ 8\ .38$	$+\ 85\ 83\ .80$	$+\ 25\ 55\ .45$
	$\log \csc \pi = \log r$	1.7596999	1.7595414	1.7593865
	ar. co. $\log r'$	9.9930339	9.9930353	9.9930367
	Constant $\log \sin \pi_0$	5.61894		
(1)	$\log b$	7.37167	7.87152	7.37136
(2)	ar. co. $\log (1 - b)$	0.001028	0.001023	0.001022
	$\log \cos \delta$	9.96805	9.96855	9.96905
	$\log \sec \delta'$	0.02973	0.02970	0.02968
	$\log (a - a')$	$n3.38263$	$n2.49511$	8.25154
	$\log (a - a')$	0.75310	9.86590	$n0.62265$
	$a - a'$	$+\quad 5''.66$	$+\quad 0''.73$	$-\quad 4''.19$
(1) + (2)		7.37269	7.37254	7.37238
	$\log (\delta - \delta')$	8.43191	8.32915	8.19185
	$\log (d - \delta')$	$n0.80460$	$n0.70169$	$n0.56423$
	$d - \delta'$	$-\quad 6''.38$	$-\quad 5''.03$	$-\quad 3''.67$
	a	$118°\ 2'\ 18''.16$	$118°\ 4'\ 43''.87$	$118°\ 7'\ 9''.58$
	d	$20\ 57\ 23\ .04$	$20\ 56\ 57\ .57$	$20\ 56\ 32\ .08$
$\log (1 - b) = \log g$		9.998977	9.998977	9.998978

II. Co-ordinates x, y, and z.

		1^h	2^h	3^h
	$a - a$	$-0°\ 40'\ 19''.06$	$-0°\ 5'\ 13''.42$	$+0°\ 29'\ 48''.77$
	$\delta - d$	$+\ 45\ 9\ .76$	$+\ 35\ 88\ .83$	$+\ 25\ 59\ .12$
	$\delta + d$	$42\ 39\ 55\ .84$	$42\ 29\ 83\ .97$	$42\ 19\ 3\ .28$
	$\log \sin (a - a)$	$n8.0692116$	$n7.1817014$	7.9381239
	$\log \cos \delta$	9.9680502	9.9685481	9.9690490
$\log r \cos \delta \sin (a - a) = \log x$		$n9.7969617$	$n8.9097909$	9.6665594
	x	-0.626559	-0.081244	0.464044
	$\log \cos^2 \tfrac{1}{2}(a - a)$	9.9999850	9.9999998	9.9999920
	$\log \sin (\delta - d)$	8.1184932	8.0157434	7.8784502
$\log (3) = \log r \sin (\delta - d) \cos^2 \tfrac{1}{2}(a - a)$		9.8781781	9.7752846	9.6878287
	$\log \sin^2 \tfrac{1}{2}(a - a)$	5.5363780	3.7613394	5.2741910
	$\log \sin (\delta + d)$	9.8310485	9.8296235	9.8281695
$\log (4) = \log r \sin (\delta + d) \sin^2 \tfrac{1}{2}(a - a)$		7.1271264	5.3405043	6.8617470
	(3)	$+0.755402$	$+0.596053$	$+0.434329$
	(4)	$+0.001340$	$+0.000022$	$+0.000727$
(3) + (4)	y	$+0.756742$	$+0.596075$	$+0.435056$
	$\log \cos (\delta - d)$	9.9999625	9.9999766	9.9999876
$\log (5) = \log r \cos (\delta - d) \cos^2 \tfrac{1}{2}(a - a)$		1.7596474	1.7595178	1.7593661
	$\log \cos (\delta + d)$	9.8664780	9.8676822	9.8688939
$\log (6) = \log r \cos (\delta + d) \sin^2 \tfrac{1}{2}(a - a)$		7.1625559	5.8885630	6.9024714
$\log [(5) - (6)] = \log z$		1.7596364	1.7595176	1.7593601

III. Log i and l, for exterior contacts. [Constant log $= 7.668803$]

	1ʰ	2ʰ	3ʰ
log $r'g$	0.005943	0.005942	0.005941
Const. $-$ log $r'g =$ log sin f	7.662860	7.662861	7.662862
log sec f	0.000005		
log k cosec f	1.772140	1.772139	1.772138
log $[z + k$ cosec $f] =$ log c	2.066963	2.066904	2.066826
log tan f $=$ log i	7.662865	7.662866	7.662867
log ic $=$ log l	9.729828	9.729770	9.729693
l	0.536819	0.536747	0.536652

Log i and l for interior contacts. [Constant log $= 7.666691$]

Const. $-$ log $r'g =$ log sin f	7.660748	7.660749	7.660750
log sec f	0.000005		
log k cosec f	1.774252	1.774251	1.774250
log $[z - k$ cosec $f] =$ log c	n0.293985	n0.297413	n0.301919
log tan f $=$ log i	7.660753	7.660754	7.660755
log ic $=$ log l	n7.954738	n7.958167	n7.962674
l	$-$0.009010	$-$0.009082	$-$0.009176

IV. The computation being made for the other hours in the same manner, the results are collected in the following tables.

	a	d	Exterior Contacts.		Interior Contacts.	
			l	log i	l	log i
0ʰ	117° 59′ 52″.44	20° 57′ 48″.50	0.536867	7.662864	$-$ 0.008960	7.660752
1	118　2 18 .16	57 23 .04	0.536819	65	0.009010	53
2	4 43 .87	56 57 .57	0.536747	66	0.009082	54
3	7　9 .58	56 32 .08	0.536652	67	0.009176	55
4	9 35 .27	56　6 .58	0.536533	68	0.009293	56
5	12　0 .95	55 41 .06	0.536391	69	0.009434	57

	x	Δ_1	Δ_2	Δ_3	y	Δ_1	Δ_2	Δ_3
0ʰ	$-$ 1.171856	$+$ 0.545297			$+$ 0.917040	$-$ 0.160298		
1	$-$ 0.626559	0.545315	$+$ 18	-45	$+$ 0.756742	$-$ 0.160667	-369	$+17$
2	$-$ 0.081244	0.545288	$-$ 27	-60	$+$ 0.596075	$-$ 0.161019	-352	$+19$
3	$+$ 0.464044	0.545201	$-$ 87	-75	$+$ 0.485056	$-$ 0.161352	-333	$+20$
4	$+$ 1.009245	0.545089	-162		$+$ 0.273704	$-$ 0.161665	-318	
5	$+$ 1.554284				$+$ 0.112039			

For the values of the hourly differences of x and y, we find from the above, by Art. 296,

	x'	$\log x'$	y'	$\log y'$
0ᴴ	0.545306	9.736640	— 0.160483	n9.205429
1	0.545315	648	— 0.160667	5927
$T_0 = 2$	0.545310	644	— 0.160846	6410
3	0.545288	626	— 0.161019	6877
4	0.545245	592	— 0.161186	7327
5	0.545176	537	— 0.161345	7756

and for any given time $T = T_0 + \tau$, we have

$$\left. \begin{array}{l} x = -\, 0.081244 + x'\tau \\ y = +\, 0.596075 + y'\tau \end{array} \right\} \quad (492)$$

Finally, to facilitate the computation of the hour angle $\mu - a = \mu_1 - \omega$ (Art. 291), we prepare the values of μ_1 for each of the Greenwich hours. Thus, for $T = 1^h$, we have

From the Ephemeris, July 18, 1860,

Sid. time at mean noon	=	7ʰ 46ᵐ 4ˢ.03	
Sid. equivalent of 1ʰ mean t.	=	1 0 9 .86	
Greenwich sid. time	=	8 46 13 .89	
" " " in arc,	=	131° 33' 28".35	
a	=	118 2 18 .16	
μ_1	=	13 31 10 .19	

Thus we form the following table, to which is also added for future use the value of the logarithm of

$\mu' =$ the hourly difference of μ_1 in parts of the radius;

	μ_1	Hourly diff.
0ᴴ	358° 31' 8".0	
1	13 31 10 .2	
2	28 31 12 .3	54002".15
3	43 31 14 .4	
4	58 31 16 .6	
5	73 31 18 .7	

$\log \mu' = \log 54002''.15 \sin 1''$
$= 9.417986$

I proceed to consider the principal problems relating to the general prediction of eclipses, in which the preceding results will be applied.

Outline of the Shadow on the Surface of the Earth.

298. *To find the outline of the moon's shadow upon the earth at a given time.*—This outline is the intersection of the cone of shadow with the earth's surface; or, it is the curve on the surface of the earth from every point of which a contact of the sun's and moon's limbs may be observed at the given time. Let

$$T = \text{the given time reckoned at the first meridian,}$$

and let a, d, x, y, l, and $\log i$ be taken from the general tables of the eclipse for this time. Then the co-ordinates ξ, η, ζ of any place at which a contact may be observed at the given time must satisfy the conditions (491),

$$\left. \begin{array}{l} (l - i\zeta) \sin Q = x - \xi \\ (l - i\zeta) \cos Q = y - \eta \end{array} \right\} \quad (493)$$

Let

$$\vartheta = \text{the hour angle of } the\ point\ Z,$$
$$\omega = \text{the west longitude of the place;}$$

then we have

$$\vartheta = \mu - a = \mu_1 - \omega$$

and the equations (483) become

$$\left. \begin{array}{l} \xi = \rho \cos \varphi' \sin \vartheta \\ \eta = \rho \sin \varphi' \cos d - \rho \cos \varphi' \sin d \cos \vartheta \\ \zeta = \rho \sin \varphi' \sin d + \rho \cos \varphi' \cos d \cos \vartheta \end{array} \right\} \quad (494)$$

The five equations in (493) and (494) involve the six variables ξ, η, ζ, φ', ϑ, and Q, any one of which may be assumed arbitrarily (excluding, of course, assumed values that give impossible or imaginary results); then for each assumed value of the arbitrary quantity we shall have five equations, which fully determine five unknown quantities, and thereby one point of the required curve. I shall take Q as the arbitrary variable.

In the present form of the equations (494), they involve the unknown quantity ρ, which being dependent upon φ' cannot be determined until the latter is found. This seems to involve the necessity of at first neglecting the compression of the earth, by putting $\rho = 1$, and after an approximate value of φ' has been found, and thereby also the value of ρ, repeating the computation. But, by a simple transformation given by BESSEL, this double computation is rendered unnecessary, and the compression of

the earth is taken into account from the beginning. If φ is the geographical latitude, we have (Art. 82)

$$\rho \cos \varphi' = \frac{\cos \varphi}{\sqrt{(1 - ee \sin^2 \varphi)}} \qquad \rho \sin \varphi' = \frac{\sin \varphi (1 - ee)}{\sqrt{(1 - ee \sin^2 \varphi)}}$$

in which

$$\log ee = 7.824409 \qquad \log \sqrt{(1 - ee)} = 9.9985458$$

If we take a new variable φ_1, such that

$$\cos \varphi_1 = \frac{\cos \varphi}{\sqrt{(1 - ee \sin^2 \varphi)}}$$

we shall have

$$\sin \varphi_1 = \sqrt{(1 - \cos^2 \varphi_1)} = \frac{\sin \varphi \sqrt{(1 - ee)}}{\sqrt{(1 - ee \sin^2 \varphi)}}$$

or

$$\cos \varphi_1 = \rho \cos \varphi'$$
$$\sqrt{(1 - ee)} \sin \varphi_1 = \rho \sin \varphi'$$
$$\tan \varphi = \frac{\tan \varphi_1}{\sqrt{(1 - ee)}}$$

Hence the equations (494) become

$$\xi = \cos \varphi_1 \sin \vartheta$$
$$\eta = \sin \varphi_1 \cos d \sqrt{(1 - ee)} - \cos \varphi_1 \sin d \cos \vartheta$$
$$\zeta = \sin \varphi_1 \sin d \sqrt{(1 - ee)} + \cos \varphi_1 \cos d \cos \vartheta$$

Put

$$\rho_1 \sin d_1 = \sin d \qquad \rho_2 \sin d_2 = \sin d \sqrt{(1 - ee)} \left.\right\}$$
$$\rho_1 \cos d_1 = \cos d \sqrt{(1 - ee)} \qquad \rho_2 \cos d_2 = \cos d \qquad \left.\right\} \quad (495)$$

The quantities ρ_1, d_1, ρ_2, d_2, may be computed for the same times as the other quantities in the tables of the eclipse, and hence obtained by interpolation for the given time. The factors ρ_1 and ρ_2 will be sensibly constant for the whole eclipse. We now have

$$\xi = \cos \varphi_1 \sin \vartheta$$
$$\eta = \rho_1 \sin \varphi_1 \cos d_1 - \rho_1 \cos \varphi_1 \sin d_1 \cos \vartheta$$
$$\zeta = \rho_2 \sin \varphi_1 \cos d_2 + \rho_2 \cos \varphi_1 \cos d_2 \cos \vartheta$$

Let us put

$$\eta_1 = \frac{\eta}{\rho_1}$$

and assume ζ_1, so that

$$\xi^2 + \eta_1^2 + \zeta_1^2 = 1 \qquad (496)$$

or, which is equivalent, let us take the system

$$\left.\begin{aligned}
\xi &= \cos \varphi_1 \sin \vartheta \\
\eta_1 &= \sin \varphi_1 \cos d_1 - \cos \varphi_1 \sin d_1 \cos \vartheta \\
\zeta_1 &= \sin \varphi_1 \sin d_1 + \cos \varphi_1 \cos d_1 \cos \vartheta
\end{aligned}\right\} \quad (497)$$

The quantity ζ_1 differs so little from ζ that we may in practice substitute one for the other in the small term $i\zeta$; but if theoretical accuracy is desired we can readily find ζ when ζ_1 is known; for the second and third of (497) give

$$\begin{aligned}
\cos \varphi_1 \cos \vartheta &= -\eta_1 \sin d_1 + \zeta_1 \cos d_1 \\
\sin \varphi_1 &= \eta_1 \cos d_1 + \zeta_1 \sin d_1
\end{aligned}$$

which substituted in the value of ζ give

$$\zeta = \rho_2 \zeta_1 \cos (d_1 - d_2) - \rho_2 \eta_1 \sin (d_1 - d_2) \qquad (498)$$

Our problem now takes the following form. We have first the three equations

$$\left.\begin{aligned}
(l - i\zeta_1) \sin Q &= x - \xi \\
(l - i\zeta_1) \cos Q &= y - \rho_1 \eta_1 \\
\xi^2 + \eta_1^2 + \zeta_1^2 &= 1
\end{aligned}\right\} \quad (499)$$

which for each assumed value of Q determine $\xi, \eta_1,$ and ζ_1. Then we have

$$\left.\begin{aligned}
\cos \varphi_1 \sin \vartheta &= \xi \\
\cos \varphi_1 \cos \vartheta &= -\eta_1 \sin d_1 + \zeta_1 \cos d_1 \\
\sin \varphi_1 &= \eta_1 \cos d_1 + \zeta_1 \sin d_1
\end{aligned}\right\} \quad (500)$$

which determine φ_1 and ϑ. Then the latitude and longitude of a point of the required outline are found by the equations

$$\tan \varphi = \frac{\tan \varphi_1}{\sqrt{(1 - ee)}} \qquad\qquad \omega = \mu_1 - \vartheta \qquad (501)$$

To solve (499), let β and γ be found by the equations

$$\left.\begin{aligned}
\sin \beta \sin \gamma &= x - l \sin Q = a \\
\sin \beta \cos \gamma &= \frac{y}{\rho_1} - \frac{l \cos Q}{\rho_1} = b
\end{aligned}\right\} \quad (502)$$

then we have

$$\begin{aligned}
\xi &= \sin \beta \sin \gamma + i\zeta_1 \sin Q \\
\eta_1 &= \sin \beta \cos \gamma + i\zeta_1 \cos Q
\end{aligned}$$

where we have omitted ρ_1 as a divisor of the small term $i\zeta_1 \cos Q$, since we have very nearly $\rho_1 = 1$. Substituting these values in the last equation of (499), we find

$$\zeta_1^2 = \cos^2 \beta - 2i\zeta_1 \sin \beta \cos (Q - \gamma) - (i\zeta_1)^2$$

Neglecting the terms involving i^2 as practically insensible, this gives

$$\zeta_1 = \pm [\cos \beta - i \sin \beta \cos (Q - \gamma)]$$

In order to remove the ambiguity of the double sign, let us put

$$Z = \text{the zenith distance of } \textit{the point } Z \, (\text{Art. 289});$$

then, since $\vartheta = \mu - a$ is the hour angle of this point, we have

$$\cos Z = \sin \varphi \sin d + \cos \varphi \cos d \cos \vartheta$$

which by means of the preceding equations is reduced to

$$\cos Z = \zeta_1 \rho_1 \frac{\sin \varphi}{\sin \varphi_1} \qquad (503)$$

Hence $\cos Z$ and ζ_1 have the same sign.

But, in order that the eclipse may be *visible* from a point on the earth's surface, we must, in general, have Z less than 90°; that is, $\cos Z$ must be positive, and therefore ζ_1 must be taken only with the positive sign. The negative sign would give a second point on the surface of the earth from which, if the earth were not opaque, the same phase of the eclipse would also be observed at the given time. In fact, every element of the cone of shadow which intersects the earth's surface at all, intersects it in two points, and our solution gives both points.

If we put

$$\epsilon = \frac{i \cos (Q - \gamma)}{\sin 1''} \qquad (504)$$

we have

$$\zeta_1 = \cos \beta - \sin \beta \sin \epsilon$$

or, with sufficient accuracy,

$$\zeta_1 = \cos (\beta + \epsilon) \qquad (505)$$

Thus, β and γ being determined by (502), ζ_1 is determined by (504) and (505): hence also ξ and η_1 by the equations

$$\left. \begin{array}{l} \xi = a + i\zeta_1 \sin Q \\ \eta_1 = b + i\zeta_1 \cos Q \end{array} \right\} (506)$$

The problem is, therefore, fully resolved; but, for the convenience of logarithmic computation, let c and C be determined by the equations

$$\left.\begin{array}{l} c \sin C = \eta_1 \\ c \cos C = \zeta_1 \end{array}\right\} (507)$$

then the equations (500) become

$$\left.\begin{array}{l} \cos \varphi_1 \sin \vartheta = \xi \\ \cos \varphi_1 \cos \vartheta = c \cos (C + d_1) \\ \sin \varphi_1 = c \sin (C + d_1) \end{array}\right\} (508)$$

The curve thus determined will be the intersection of the penumbral cone, or that of the umbral cone, with the earth's surface, according as we employ the value of l for the one or the other.

299. The above solution is direct, though theoretically but approximate, since we have neglected terms of the order of i^2. It can, however, readily be made quite exact as follows. We have, by substituting the values of ζ_1 and η_1 in (498), and neglecting the term involving the product $i \sin (d_1 - d_2)$, which is of the same order as i^2,

$$\zeta = \rho_2 \cos (\beta + \epsilon) - \rho_2 \sin \beta \cos \gamma \sin (d_1 - d_2)$$

and, putting

$$\epsilon' = (d_1 - d_2) \cos \gamma$$

we have, within terms of the order i^2,

$$\zeta = \rho_2 \cos (\beta + \epsilon + \epsilon') \qquad (509)$$

The substitution of this value of ζ in the term $i\zeta$ involves only an error of the order i^3, which is altogether insensible. The exact solution of the problem is, therefore, as follows. Find β and γ for each assumed value of Q, by the equations

$$\sin \beta \sin \gamma = x - l \sin Q = a$$

$$\sin \beta \cos \gamma = \frac{y}{\rho_1} - \frac{l \cos Q}{\rho_1} = b$$

then ϵ and ϵ' by the equations

$$\epsilon = \frac{i \cos (Q - \gamma)}{\sin 1''} \qquad \epsilon' = (d_1 - d_2) \cos \gamma$$

Find β' and γ' by the equations

$$\sin \beta' \sin \gamma' = a + i\rho_2 \cos (\beta + \epsilon + \epsilon') \sin Q = \xi$$

$$\sin \beta' \cos \gamma' = b + \frac{i\rho_2 \cos (\beta + \epsilon + \epsilon') \cos Q}{\rho_1} = \eta_1$$

then we have, rigorously,

$$\zeta_1 = \cos \beta'$$

and these values of ξ, η_1, and ζ_1 may then be substituted in (500), which can be adapted for logarithmic computation as before.*

300. It remains to be determined whether the eclipse is begin-ning or ending at the places thus found. A point on the earth's surface which at a given time T is upon the surface of the cone of shadow will at the next consecutive instant $T + dT$ be *within* or *without* the cone according as the eclipse is *beginning* or *ending* at the time T; the former or the latter, according as the distance $\varDelta = \sqrt{[(x - \xi)^2 + (y - \eta)^2]}$ becomes at the time $T + dT$ *less* or *greater* than the radius of the shadow $l - i\zeta$. In the case of total eclipse $l - i\zeta$ is a negative quantity, but by comparing $\varDelta^2$ with $(l - i\zeta)^2$ we shall obtain the required criterion for all cases; and, therefore, the criterion of *beginning* or *ending*, either of partial or of total eclipse, will be the *negative* or *positive* value of the differential coefficient, relatively to the time, of the quantity

$$(x - \xi)^2 + (y - \eta)^2 - (l - i\zeta)^2$$

or the negative or positive value of the quantity

$$(x - \xi)\left(\frac{dx}{dT} - \frac{d\xi}{dT}\right) + (y - \eta)\left(\frac{dy}{dT} - \frac{d\eta}{dT}\right) - (l - i\zeta)\left(\frac{dl}{dT} - i\frac{d\zeta}{dT}\right)$$

---

* In this problem, as well as in most of the subsequent ones, I have not followed BESSEL's methods of solution, which, being mathematically rigorous, though as simple as such methods can possibly be, are too laborious for the practical purposes of mere prediction. As a refined and exhaustive disquisition upon the whole theory, BESSEL's *Analyse der Finsternisse*, in his *Astronomische Untersuchungen*, stands alone. On the other hand, the approximate solutions heretofore in common use are mostly quite imperfect; the compression of the earth, as well as the augmentation of the moon's semidiameter, being neglected, or only taken into account by repeating the *whole* computation, which renders them as laborious as a rigorous and direct method. I have endeavored to remedy this, by so arranging the successive approximations, when these are necessary, that only a small part of the whole computation is to be repeated, and by taking the compression of the earth into account, in all cases, from the commencement of the computation. In this manner, even the first approxima-tions by my method are rendered more accurate than the common methods.

where we omit the insensible variation of i. For brevity, let us write x', y', &c. for $\dfrac{dx}{dT}$, $\dfrac{dy}{dT}$, &c. and denote the above quantity by P; then, after substituting the values of $x - \xi = (l - i\zeta) \sin Q$, $y - \eta = (l - i\zeta) \cos Q$, we have

$$P = L \left[(x' - \xi') \sin Q + (y' - \eta') \cos Q - (l' - i\zeta') \right]$$

in which $L = l - i\zeta$. If we put

$$P' = (x' - \xi') \sin Q + (y' - \eta') \cos Q - (l' - i\zeta') \qquad (510)$$

we shall have

$$P = LP'$$

The quantity P will be positive or negative according as L and P' have like signs or different signs.

For exterior contacts, and for interior contacts in annular eclipse, L is positive (Art. 293), and hence for these cases the eclipse is *beginning* or *ending* according as P' is *negative* or *positive*; but for total eclipse, L being negative, we have *beginning* or *ending* according as P' is *positive* or *negative*.

We must now develop the quantity P'. Taking one hour as the unit of time, x', y', l', ξ', η', ζ', will denote the hourly changes of the several quantities. The first three of these may be derived from the general tables of the eclipse for the given time; but ξ', η', ζ' are obtained by differentiating the equations (494), in which the latitude and longitude of the point on the earth's surface are to be taken as constant. Since $\vartheta = \mu_1 - \omega$, we shall have $\dfrac{d\vartheta}{dT} = \dfrac{d\mu_1}{dT}$; and hence, putting

$$\mu' = \frac{d\mu_1}{dT} \sin 1'' \qquad\qquad d' = \frac{dd}{dT} \sin 1''$$

we find

$$\xi' = \mu' \rho \cos \varphi' \cos \vartheta = \mu' (- \eta \sin d + \zeta \cos d)$$
$$= \mu' [- y \sin d + \zeta \cos d + (l - i\zeta) \sin d \cos Q]$$
$$\eta' = \mu' \xi \sin d - d' \zeta$$
$$= \mu' [x \sin d - (l - i\zeta) \sin d \sin Q] - d' \zeta$$
$$\zeta' = - \mu' \xi \cos d + d' \eta$$
$$= \mu' [- x \cos d + (l - i\zeta) \cos d \sin Q] + d' [y - (l - i\zeta) \cos Q]$$

Substituting these values in (510), and neglecting terms involving i^2 and id' as insensible, we have

$$P' = a' - b' \cos Q + c' \sin Q - \zeta \, (\mu' \cos d \sin Q - d' \cos Q)$$

in which a', b', and c', denote the following quantities:

$$\left.\begin{aligned} a' &= -\, l' - \mu' \, ix \cos d \\ b' &= -\, y' + \mu' \, x \sin d \\ c' &= \quad x' + \mu' \, y \sin d + \mu' \, il \cos d \end{aligned}\right\} \text{(511)}$$

The values of these quantities may be computed for the same times as the other quantities in the eclipse tables, and their values for any given time will then be readily found by interpolation. For any assumed value of Q, therefore, and with the value of ζ found by (509), the value of P' may be computed, and its sign will determine whether the eclipse is beginning or ending. In most cases, a mere inspection of the tabulated values of a', b', and c', combined with a consideration of the value of Q, will suffice to determine the sign of P'; but when the place is near the northern or southern limits of the shadow, an accurate computation of P' will be necessary; and, since other applications of this quantity will be made hereafter, it will be proper to give it a more convenient form for logarithmic computation. Put

$$\left.\begin{aligned} e \sin E &= b' &\qquad f \sin F &= d' \\ e \cos E &= c' &\qquad f \cos F &= \mu' \cos d \end{aligned}\right\} \text{(512)}$$

then we have

$$P' = a' + e \sin (Q - E) - \zeta f \sin (Q - F) \qquad \text{(513)}$$

Since a' and F are both very small quantities, and a very precise computation of P' will seldom be necessary when its algebraic sign is alone required, it will be sufficient in most cases to neglect these quantities, and also to put ζ_1 for ζ, and then we shall have the following simple criterion for the case of partial or annular eclipse:

> If $\ e \sin (Q - E) < \zeta_1 f \sin Q$, the eclipse is beginning.
> If $\ e \sin (Q - E) > \zeta_1 f \sin Q$, the eclipse is ending.

For total eclipse, reverse these conditions.

301. In order to facilitate the application of the preceding as well as the subsequent problems, it is expedient to prepare the values of d_1, $\log \rho_1$, d_2, $\log \rho_2$, a', b', c', e, E, f, F, and to arrange them in tables.

For our example of the eclipse of July 18, 1860, with the values of d given on p. 454, we form the following table by the equations (495):

	d_1	$\log \rho_1$	d_2	$\log \rho_2$
0ʰ	21° 1′ 39″.5	9.9987324	20° 53′ 58″.0	9.9998143
1	1 14 .0	23	53 32 .6	45
2	0 48 .5	22	53 7 .3	46
3	0 22 .9	21	52 41 .8	47
4	20 59 57 .4	20	52 16 .4	48
5	59 31 .8	19	51 50 .9	50

The values of x', y', and l', required in (511), derived also from the eclipse tables on p. 454, by the method of Art. 75, are as follows:

	x'	y'	l'
0ʰ	+ 0.545277	— 0.160108	— 0.000038
1	5312	0486	061
2	5310	0846	084
3	5256	1188	107
4	5134	1512	130
5	4928	1818	154

Hence, by (511) we find the values of a', b', $\overset{\cdot}{c}'$ to be as follows. The values for interior contacts are seldom required.

	For exterior contacts.			For interior contacts.		
	a'	b'	c'	a'	b'	c'
0ʰ	+ 0.001356	+ 0.050342	+ 0.631779	+ 0.001350	+ 0.050342	+ 0.631165
1	+ 0.000766	+ 0.101816	+ 0.616776	+ 0.000762	+ 0.101816	+ 0.616162
2	+ 0.000175	+ 0.153241	+ 0.601711	+ 0.000175	+ 0.153241	+ 0.601097
3	— 0.000415	+ 0.204612	+ 0.586571	— 0.000413	+ 0.204612	+ 0.585957
4	— 0.001005	+ 0.255925	+ 0.571342	— 0.001000	+ 0.255925	+ 0.570728
5	— 0.001595	+ 0.307171	+ 0.556010	— 0.001586	+ 0.307171	+ 0.555395

The values of e, E, f, F, for *exterior contacts*, deduced from these values of b' and c', and from $d' = — 25''.5 \sin 1''$, by (512), are as follows:

	E	$\log e$	F	$\log f$
0^h	4° 33' 21"	9.801939	— 0° 1' 44"	9.388244
1	9 22 25	.795965	"	264
2	14 17 17	.793034	"	285
3	19 13 48	.793255	"	305
4	24 7 46	.796604	"	326
5	28 55 7	,.802923	"	347

302. To illustrate the preceding formulæ, let us find some points of the outline of the penumbra on the earth's surface at the time $T = 2^h 8^m 12^s$. For this time, we have

$$x = -0.00672 \qquad \log \rho_1 = 9.99873 \qquad \log i = 7.66287$$
$$y = +0.57409 \qquad d_1 = 20° \; 0' \; 45''$$
$$l = +0.53673 \qquad \mu_1 = 30 \; 34 \; 13$$

Let us find the points for $Q = 50°$ and $Q = 300°$. The computation may be arranged as follows:

	Q	50°	300°
By (502):	$a = \sin \beta \sin \gamma$	— 0.41788	+ 0.45815
	$b = \sin \beta \cos \gamma$	+ 0.22975	+ 0.30662
	γ	— 61° 11' 52"	56° 12' 27"
	β	28 28 52	33 27 20
Hence by (504):	ε	— 5 43	— 6 59
	$\beta + \varepsilon$	28 23 9	33 20 21
By (505): $\log \zeta_1 = \log \cos (\beta + \varepsilon)$		9.94437	9.92191
	$i\zeta_1 \sin Q$	+ 0.00310	— 0.00333
	$i\zeta_1 \cos Q$	+ 0.00260	+ 0.00192
By (506):	ξ	— 0.41478	+ 0.45482
	η_1	+ 0.23235	+ 0.30854
By (507):	$\log \eta_1 = \log c \sin C$	9.36614	9.48931
	$\log \zeta_1 = \log c \cos C$	9.94437	9.92191
	$\log c$	9.95901	9.94967
	C	14° 47' 39"	20° 16' 12"
	$C + d_1$	35 48 24	41 16 57
By (508): $\log \xi = \log \cos \varphi_1 \sin \vartheta$		$n9.61782$	9.65784
$\log c \cos (C + d_1) = \log \cos \varphi_1 \cos \vartheta$		9.86803	9.82558
	$\log \tan \vartheta$	$n9.74979$	9.83232
	$\log \cos \varphi_1$	9.92764	9.90805
$\log c \sin (C + d_1) =$	$\log \sin \varphi_1$	9.72620	9.76906
	$\log \tan \varphi_1$	9.79856	9.86101
	$\log \sqrt{(1 - ee)}$	9.99855	9.99855
	$\log \tan \varphi$	9.80001	9.86246
	ϑ	— 29° 20' 20"	34° 12' 15"
	$\mu_1 - \vartheta = \omega$	59 54 33	356 21 58
	φ	32 15 33	36 4 30

To find whether the eclipse is beginning or ending at these places, we have, from the table on p. 465, for $T = 2^h 8^m 12^s$,

$\log e$	9.7931	
E	14° 58′	
$Q - E$	35 2	285° 2′
$\log e \sin (Q - E)$	9.5521	$n9.7780$
$\log f$	9.3883	
$\log \zeta_1 f \sin Q$	9.2170	$n9.2477$

At the first point, therefore, we have $e \sin (Q - E) > \zeta_1 f \sin Q$, and the eclipse is ending. At the second point, we have $e \sin (Q - E) < \zeta_1 f \sin Q$, and the eclipse is beginning.

Rising and Setting Limits.

303. *To find the rising and setting limits of the eclipse.*—By these limits we mean the curves upon which are situated all those points of the earth's surface where the eclipse begins or ends with the sun in the horizon. It will be quite sufficient for all practical purposes to determine these limits by the condition that *the point Z is in the horizon.* This gives in (503) $\cos Z = 0$, or $\zeta_1 = 0$, and, consequently, by (496), we have

$$\xi^2 + \eta_1^2 = 1 \tag{514}$$

as the condition which the co-ordinates of the required points must satisfy.

Now, let it be required to find the place where this equation is satisfied at a given time T. Let x and y be taken for this time, then we have, by putting $\zeta_1 = 0$ in (499),

$$l \sin Q = x - \xi$$
$$l \cos Q = y - \eta$$

Let

$$
\begin{matrix}
m \sin M = x & \qquad p \sin \gamma = \xi \\
m \cos M = y & \qquad p \cos \gamma = \eta
\end{matrix}
\biggr\} \tag{515}
$$

then, from the equations

$$
\begin{matrix}
l \sin Q = m \sin M - p \sin \gamma \\
l \cos Q = m \cos M - p \cos \gamma
\end{matrix}
\biggr\} \tag{516}
$$

we deduce, by adding their squares,

$$l^2 = m^2 - 2mp \cos (M - \gamma) + p^2$$

$$2 \sin^2 \tfrac{1}{2} (M - \gamma) = 1 - \cos (M - \gamma) = \frac{l^2 - (m - p)^2}{2mp}$$

If then we put $\lambda = M - \gamma$, we have

$$\sin \tfrac{1}{2}\lambda = \pm \sqrt{\left[\frac{(l + m - p)(l - m + p)}{4mp}\right]}$$

$$\gamma = M \pm \lambda$$ } (517)

in which $\tfrac{1}{2}\lambda$ may always be taken less than 90°, but the double sign must be used to obtain the two points on the surface of the earth which satisfy the conditions at the given time.

In this formula, m, M, and l are accurately known for the given time, but p is unknown. It is evident, however, from (514) and (515), that we have nearly $p = 1$, and this value may be used in (517) for a first approximation. To obtain a more correct value of γ, let us put $\xi = \sin \gamma'$; then, by (514), we have $\eta_1 = \cos \gamma'$, and, consequently, since $\eta = \rho_1 \eta_1$,

$$p \sin \gamma = \sin \gamma'$$
$$p \cos \gamma = \rho_1 \cos \gamma'$$

Hence we have

$$\tan \gamma' = \rho_1 \tan \gamma$$

$$p = \frac{\sin \gamma'}{\sin \gamma} = \frac{\rho_1 \cos \gamma'}{\cos \gamma}$$ } (518)

and with this value of p the second computation of (517) will give a very exact value of γ. With this second value of γ a still more correct value of p could be found; but the second approximation is always sufficient.

With the second value of γ, therefore, we find the final value of γ' by the formula

$$\tan \gamma' = \rho_1 \tan \gamma$$

and then, substituting the values $\xi = \sin \gamma'$, $\eta_1 = \cos \gamma'$, $\zeta_1 = 0$, in (500), we have, for finding the latitude and longitude of the required points, the formulæ

$$\cos \varphi_1 \sin \vartheta = \sin \gamma'$$
$$\cos \varphi_1 \cos \vartheta = -\cos \gamma' \sin d_1$$
$$\sin \varphi_1 = \cos \gamma' \cos d_1$$

$$\omega = \mu_1 - \vartheta \qquad \tan \varphi = \frac{\tan \varphi_1}{\sqrt{(1 - ee)}}$$ } (519)

In the second approximation, we must compute λ and γ by (517) separately for each place.

304. The sun is rising or setting at the given time at the places thus determined, according as ϑ (which is the hour angle of the point Z) is between 180° and 360° or between 0° and 180°.

To determine whether the eclipse is beginning or ending, we may have recourse to the sign of P' (513); and it will usually be sufficient for the present problem to put both a' and $\zeta = 0$ in that expression, and then the eclipse is beginning or ending according as sin $(Q - E)$ is negative or positive. Now, by (516), we find

$$l \sin (Q - E) = m \sin (M - E) - p \sin (\gamma - E)$$

Hence, for points in the rising or setting limits,

If $m \sin (M - E) < p \sin (\gamma - E)$, the eclipse is beginning,
If $m \sin (M - E) > p \sin (\gamma - E)$, the eclipse is ending.

305. In order to apply the preceding method of determining the rising and setting limits, it is necessary first to find the extreme times between which the time T is to be assumed, or those limits of T between which the solution is possible. The two solutions given by (517) must reduce to a single one when the surface of the cone of shadow has but a single point in common with the earth's surface,—i.e. in the case of tangency of the cone and the terrestrial spheroid. Now, the two solutions reduce to one only when $\lambda = 0$, and both values of γ become $= M$; but if $\lambda = 0$, the numerator of the value of sin $\frac{1}{2}\lambda$ must also be zero; and hence the points of contact are determined by the conditions

$$l + m - p = 0 \qquad \text{and} \qquad l - m + p = 0$$

or by the conditions

$$m = p + l \qquad \text{and} \qquad m = p - l$$

There may be four cases of contact, two of exterior and two of interior contact. The two exterior contacts are the first and last, or *the beginning and the end of the eclipse generally;* the axis of the shadow is then without the earth, and therefore we must have for these cases $m = \sqrt{x^2 + y^2} = p + l$.

The first-interior contact corresponds to the last point on the earth's surface where the eclipse ends at sunrise; the second, to the first point where it begins at sunset. But these interior

contacts can occur only when the whole of the shadow on the principal plane falls within the earth, and for these cases, therefore, we must have $m = p - l$.

For the beginning and end generally we have, therefore, by (515),

$$(p + l) \sin M = x$$
$$(p + l) \cos M = y$$

Let T be the time when these conditions are satisfied, and put

$$T = T_0 + \tau$$

in which T_0 is the epoch of the eclipse tables, for which the values of x and y are x_0 and y_0. Then, x' and y' being the mean hourly changes of x and y for the time T, we have

$$x = x_0 + \tau x'$$
$$y = y_0 + \tau y'$$

Putting

$$\left. \begin{array}{ll} m_0 \sin M_0 = x_0 & n \sin N = x' \\ m_0 \cos M_0 = y_0 & n \cos N = y' \end{array} \right\} \quad (520)$$

the above conditions become

$$(p + l) \sin M = m_0 \sin M_0 + \tau \cdot n \sin N$$
$$(p + l) \cos M = m_0 \cos M_0 + \tau \cdot n \cos N$$

whence

$$(p + l) \sin (M - N) = m_0 \sin (M_0 - N)$$
$$(p + l) \cos (M - N) = m_0 \cos (M_0 - N) + n\tau$$

so that, if we put $M - N = \psi$, we have

$$\left. \begin{array}{l} \sin \psi = \dfrac{m_0 \sin (M_0 - N)}{p + l} \\[3mm] \tau = \dfrac{p + l}{n} \cos \psi - \dfrac{m_0}{n} \cos (M_0 - N) \\[3mm] T = T_0 + \tau \end{array} \right\} \quad (521)$$

in which $\cos \psi$ may be taken with either the negative or the positive sign; and it is evident that the first will give the beginning and the second the end of the eclipse generally.

For the two interior contacts we have

$$\left. \begin{array}{l} \sin \psi = \dfrac{m_0 \sin (M_0 - N)}{p - l} \\[3mm] \tau = \dfrac{p - l}{n} \cos \psi - \dfrac{m_0}{n} \cos (M - N) \end{array} \right\} \quad (522)$$

These interior contacts cannot occur when $p - l$ is less than $m_0 \sin(M_0 - N)$, which would give impossible values of $\sin \psi$.

In these formulæ we at first assume $p = 1$, and, after finding an approximate value of ψ, we have, by (517), in which $\lambda = 0$, $\gamma = M$, and in the present problem $M = N + \psi$: therefore

$$\gamma = N + \psi \qquad (523)$$

with which p is found by (518), and the second computation of (521) or (522) will then give the required times. We must employ in (523) the two values of ψ found by taking $\cos \psi$ with the positive and the negative sign; and therefore different values of p will be found for beginning and ending, so that in the second approximation separate computations will be necessary for the two cases.

In the first approximation the mean values of x', y', and l may be used, or those for the middle of the eclipse. With the approximate values of τ thus found, the true values of x', y', and l for the time $T = T_0 + \tau$ may be taken for the second approximation.

After finding the corrected value of ψ, we then have also the true value of $\gamma = N + \psi$ for each point, and hence also the true value of γ' by (518), with which the latitude and longitude of the points will be computed by (519). For the local apparent time of the phenomenon at each place we may take the value of ϑ in time, which is very nearly the sun's hour angle.

306. When the interior contacts exist, the rising and setting limits form two distinct enclosed curves on the earth's surface. If we denote the times of beginning and ending generally, determined by (521), by T_1 and T_2, and the times of interior contact, determined by (522), by T_1' and T_2', a series of points on the rising limit will be found by Art. 303, for a series of times assumed between T_1 and T_1', and points of the setting limit for times assumed between T_2' and T_2.

When the interior contacts do not exist, the rising and setting limits meet and form a single curve extending through the whole eclipse. The form of this curve may be compared to that of the figure 8 much distorted. A series of points upon it will be found by assuming times between T_1 and T_2.

307. EXAMPLE.—Let us find the rising and setting limits of the eclipse of July 18, 1860.

For a second approximation, therefore, recomputing (521), we now find

log sin ψ	n9.55316	n9.55269
log cos ψ	n9.97032	9.97039
T	23ʰ.9099	4ʰ.9587
ψ	200° 56′ 27″	339° 4′ 58″
$N + \psi = \gamma$	307 20 24	85 34 4
and by (518): log tan γ′	n0.11626	1.10936

Then, for the latitude and longitude of the points, we have, by (519),

d_1	21° 1′ 41″	20° 59′ 33″
μ_1	357 10 2	72 54 8
ϑ	254 38 52	91 35 43
$\mu_1 - \vartheta = \omega$	102 31 10	341 18 25
φ	34 38 40	4 9 49

Therefore the eclipse begins on the earth generally on July 17, 23ʰ 54ᵐ.6 Greenwich mean time, in west longitude 102° 31′ 10″ and latitude 34° 38′ 40″, and ends July 18, 4ʰ 57ᵐ.5 in longitude 341° 18′ 25″ and latitude 4° 9′ 49″.

It is evident that for practical purposes the first approximation, which gives the times within half a minute, is quite sufficient, especially since the effect of refraction has not yet been taken into account. (See Art. 327.)

Secondly.—We now pass to the computation of the curve which contains all the points where the eclipse begins or ends at sunrise or sunset. In the present example, this curve extends through the whole eclipse, since we have $m_0 \sin (M_0 - N) > 1 - l$: hence the required points will be found for Greenwich times assumed between July 17, 23ʰ.91 and July 18, 4ʰ.96. Let us take the series

$$T, \; 0^h, \; 0^h.2, \; 0^h.4, \; 0^h.6, \; 0^h.8 \ldots\ldots 4^h.6, \; 4^h.8$$

The computation being carried on for all the points at once, the regular progression of the corresponding numbers for the successive times furnishes at each step a verification of its correctness. To illustrate the use of the formulæ, I give the computation for $T = 2^h.0$ nearly in full. For this time, we find, from p. 454 and p. 464,

$$x = m \sin M = -\,0.08124 \qquad l = 0.53675 \qquad d_1 = 21° 0′ 49″$$
$$y = m \cos M = +\,0.59608 \qquad\qquad\qquad \log \rho_1 = 9.99873$$

SOLAR ECLIPSE, July 18, 1860.—RISING AND SETTING LIMITS.—(*Continued.*)

Greenwich Mean Time.	Latitude. ϕ	Long. W. from Greenwich. ω	Local App. Time. ϑ		
3ʰ.0	+ 62° 43′	265° 37′	9ʰ 11ᵐ.6	Ends at Sunset.	
.2	58 44	277 27	8 36 .3	"	"
.4	54 42	286 49	8 10 .8	"	"
.6	50 35	294 47	7 51 .0	"	"
.8	46 21	301 53	7 34 .6	"	"
4 .0	41 55	308 26	7 20 .3	"	"
2	37 10	314 40	7 7 .4	"	"
.4	31 57	320 43	6 55 .2	"	"
.6	25 55	326 48	6 42 .9	"	"
.8	18 11	333 18	6 28 .9	"	"

These points being projected upon a chart (see p. 504), the whole curve may be accurately traced through them. It will be seen that the method of assuming a series of equidistant times gives more points in those portions of the curve where the curvature is greatest than in other portions, thus facilitating the accurate delineation of the curve. This advantage appears to have been overlooked by those who have preferred methods (such, for example, as HANSEN's) in which a series of equidistant latitudes is assumed.

308. The preceding computations have been made for the penumbra; but we may employ the same method to determine the rising and setting limits of total or annular eclipse by employing in the formulæ the value of l for interior contacts. These limits, however, embrace so small a portion of the earth's surface that they are practically of little interest.

Curve of Maximum in the Horizon.

309. *To find the curve on which the maximum of the eclipse is seen at sunrise or sunset.*—When a point of the earth's surface whose co-ordinates are ξ, η, and ζ is not on the surface of the cone of shadow, but at a distance $\varDelta$ from the axis of the cone, we have the conditions (485),

$$\left. \begin{aligned} \varDelta \sin Q &= x - \xi \\ \varDelta \cos Q &= y - \eta \end{aligned} \right\} \quad (524)$$

Hence, for any given time, the conditions (524) become

$$\pm \varDelta \sin E = x - \xi$$
$$\pm \varDelta \cos E = y - \eta$$

which with the condition

$$\xi^2 + \eta_1^2 = 1$$

must determine the required points of our curve. The angle E is here known for the given time, being directly obtained from its tabulated values, but $\varDelta$ is unknown. Putting, as in the preceding problem,

$$m \sin M = x \qquad p \sin \gamma = \xi$$
$$m \cos M = y \qquad p \cos \gamma = \eta$$

we have

$$\pm \varDelta \sin E = m \sin M - p \sin \gamma$$
$$\pm \varDelta \cos E = m \cos M - p \cos \gamma$$

whence

$$0 = m \sin (M - E) - p \sin (\gamma - E)$$
$$\pm \varDelta = m \cos (M - E) - p \cos (\gamma - E)$$

Therefore, putting $\psi = \gamma - E$, we have

$$\left. \begin{aligned} \sin \psi &= \frac{m \sin (M - E)}{p} \\[2mm] \pm \varDelta &= m \cos (M - E) - p \cos \psi \end{aligned} \right\} \quad (527)$$

The first of these equations will give two values of ψ, since we may take $\cos \psi$ with the positive or the negative sign; but, as only those places satisfy the problem which are actually *within* the shadow, we must have $\varDelta < l$, or, at least, $\varDelta$ not greater than l. That value of ψ which would give $\varDelta > l$ must, therefore, be excluded: so that in general we shall have at a given time but one solution.

It will be quite accurate enough, considering the degree of precision above assigned, to employ in (527) a mean value of p, or, since p falls between ρ_1 and unity, to take $\log p = \frac{1}{2} \log \rho_1$. But, if we wish a more correct value, we have only to take

$$\gamma = \psi + E \qquad (528)$$

and then find p as in (518); after which (527) must be recomputed.

Having found the true value of ψ by (527), and of γ by (528), we then have γ' by the equation

$$\tan \gamma' = \rho_1 \tan \gamma$$

and the latitude and longitude of each point of the curve by (519).

The limiting times between which the solution is possible will be known from the computation of the rising and setting limits, in which we have already employed the quantity $m \sin (M - E)$; and the present curve will be computed only for those times for which $m \sin (M - E) < l$. These limiting times are also the same as those for the northern and southern limiting curves, which will be determined in Art. 313.

310. The degree of obscuration is usually expressed by the fraction of the sun's apparent diameter which is covered by the moon's disc. When the place is so far immersed in the penumbra as to be on the edge of the total shadow, the obscuration is total; in this case the distance of the place from the edge of the penumbra is equal to the absolute difference of the radii of the penumbra and the umbra, that is, to the algebraic sum $L + L_1$, L_1 denoting the radius of the umbra (which is, by Art. 293, negative); but in any other case the distance of the place within the penumbra is $L - \varDelta$: hence, if D denotes the degree of obscuration expressed as a fraction of the sun's apparent diameter, we shall have, very nearly,

$$D = \frac{L - \varDelta}{L + L_1} \qquad (529)$$

This formula may also be used when the eclipse is annular, in which case L_1 is essentially positive; and even when $\varDelta$ is zero, and the eclipse consequently central, the value of D given by the formula will be less than unity, as it should be, since in that case there is no total obscuration.

In the present problem we have

$$D = \frac{l - \varDelta}{l + l_1} \qquad (529^*)$$

in which l and l_1 are the radii of the penumbra and umbra on the principal plane, as found by (488).

EXAMPLE.—In the eclipse of July 18, 1860, compute the curve on which the maximum of the eclipse is seen in the horizon.

In the computation of the rising and setting limits, the quantity $m \sin(M - E)$ was less than unity only from $T = 0^h.6$ to $T = 4^h.2$: so that the present curve may be computed for the series of times $0^h.6$, $0^h.8 \ldots \ldots 4^h.0$, $4^h.2$. For an approximate computation we may take $\log p = \frac{1}{2} \log \rho_1 = 9.9994$, and employ only four decimal places in the logarithms throughout.

The computation for $T = 2^h$ is as follows. For this time we have already found (p. 473)

$\log m$	9.7793
M	$352° \ 14'.4$
E	$14 \ \ 17.3$
$M - E$	$337 \ \ 57.1$
$\log m \sin(M - E)$	$n9.3538$
$\log p$	9.9994
$\log \sin \psi$	$n9.3544$
$\log \cos \psi$	9.9886
$\log p \cos \psi$	9.9880
$\log m \cos(M - E)$	9.7463
$m \cos(M - E)$	$+ 0.5575$
$p \cos \psi$	$+ 0.9727$
Δ	0.4152

Hence, by (527),

Here, if $\cos \psi$ were taken with the negative sign we should find $\Delta = 1.5302$, which is greater than l. Taking it, therefore, with the positive sign only, we have

ψ	$- 13° \ \ 4'.3$
$\psi + E = \gamma$	$+ \ \ 1 \ \ 13.$
$\log \tan \gamma$	8.3271
$\log \tan \gamma'$	8.3258

$\log \rho_1 = 9.9987$

with which we find, by (519),

ϑ	$176° \ 37'.2$
μ_1	$28 \ \ 31.2$
ω	$211 \ \ 54$
φ	$69 \ \ 1$
App. time $= \vartheta$ in time	$11^h \ 46^m.5$
	Sunset.

To express the degree of obscuration according to (529*) we have, taking the mean values of l and l_1 (p. 454),

$$l = \quad 0.5366 \qquad l - \Delta = 0.1214$$
$$l_1 = - \ 0.0092 \qquad D = \frac{0.1214}{0.5274} = 0.23$$
$$l + l_1 = \quad 0.5274$$

In the same manner all the following results are obtained:

For any given time T, therefore, we are to find that point of the outline of the shadow on the surface of the earth for which the value of Q and its corresponding ζ satisfy this equation. This can be effected only indirectly, or by successive approximations. For this purpose, we must know at the outset an approximate value of Q; and therefore, before proceeding any further, we must show how such an approximate value may be found.

We can readily determine sufficiently narrow limits between which Q may be assumed. For this purpose, neglecting a' in (530), as well as F, which are always very small, we have, approximately,

$$e \sin (Q - E) = \zeta f \sin Q$$

The extreme values of ζ are $\zeta = 0$ and $\zeta = 1$. The first gives $\sin (Q - E) = 0$, and therefore for a first limit we have

$$Q = E \qquad \text{or} \qquad Q = 180° + E$$

The second gives

$$e \sin (Q - E) = f \sin Q$$

whence

$$\tan (Q - \tfrac{1}{2} E) = \frac{e + f}{e - f} \tan \tfrac{1}{2} E$$

Put

$$\tan \psi = \frac{e + f}{e - f} \tan \tfrac{1}{2} E$$

then the equation $\tan (Q - \tfrac{1}{2} E) = \tan \psi$ gives for our second limits

$$Q = \tfrac{1}{2} E + \psi \qquad \text{or} \qquad Q = 180° + \tfrac{1}{2} E + \psi$$

To compute ψ readily, put

$$\tan \nu = \frac{f}{e}$$

then

$$\tan \psi = \tan (45° + \nu) \tan \tfrac{1}{2} E$$

$$\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \quad (531)$$

and Q is to be assumed

between E and $\tfrac{1}{2} E + \psi$
or between $180° + E$ and $180° + \tfrac{1}{2} E + \psi$

These limits may be computed in advance for the principal hours of the eclipse from the previously tabulated values of E, e, and f, and an approximate value of Q may then be easily inferred for a given time with sufficient precision for a first approximation.

When the shadow passes wholly within the earth, there are two limiting curves, northern and southern. For one of these Q is to be taken between E and $\frac{1}{2} E + \psi$; for the other, between $180° + E$ and $180° + \frac{1}{2} E + \psi$. Since E is always an acute angle, positive or negative, it follows that when Q is taken between E and $\frac{1}{2} E + \psi$, its cosine is in general positive, while it is negative in the other case. The equation $\eta = y - (l - i\zeta) \cos Q$ shows that η will be less in the first case and greater in the second, and hence *the values of Q between E and $\frac{1}{2} E + \psi$ belong to the southern limit,* and *the values of Q between $180° + E$ and $180° + \frac{1}{2} E + \psi$ belong to the northern limit.*

There is only one limit, northern or southern, when one of the series of values of Q would give impossible values of ζ in the computation of the outline of the shadow by Art. 298. But when the rising and setting limits have been determined, the question of the existence of one or both of the northern and southern limits is already settled; for if the rising and setting limits extend through the whole eclipse in north latitude, only the southern limiting curve of our present problem exists, and *vice versa;* while if the rising and setting limits form two distinct curves, we have both a northern and southern limiting curve; and the latter must evidently connect the extreme northern and southern points respectively of the two enclosed rising and setting curves. In our example of the eclipse of July 18, 1860, there exists only the southern limiting curve of the present problem, the penumbral shadow passing over and beyond the north pole of the earth.

Having assumed a value of Q, we find ζ_1 by the equations (502), (504) and (505), and then ζ by (509). This computed value of ζ and the assumed value of Q being substituted in (530), this equation will be satisfied only when the true value of Q has been assumed. To find the correction of Q, let us suppose that when the equation has been computed logarithmically we find

$$\log \zeta f \sin (Q - F) - \log [a' + e \sin (Q - E)] = \varkappa$$

If then dQ and $d\zeta$ are the corrections which Q and ζ require in

order to reduce x to zero, we have, by differentiating this equation,

$$\left[\cot(Q-F)-\frac{e\cos(Q-E)}{a'+e\sin(Q-E)}\right]\frac{dQ}{A}+\frac{d\zeta}{A\zeta}=-x$$

in which A is the reciprocal of the modulus of common logarithms.

In this differential equation we may neglect a' without sensibly affecting the rate of approximation. If then we put

$$g=-\frac{d\zeta}{\zeta\,dQ}$$

we shall have

$$dQ=\frac{Ax}{\cot(Q-E)-\cot(Q-F)+g}$$

This value of dQ is yet to be reduced to seconds by multiplying it by cosec $1''$ or $206265''$.

To find g, we may take, as a sufficiently exact expression for computing dQ,

$$g=-\frac{d\zeta_1}{\zeta_1\,dQ}$$

and by differentiating (502) (omitting the factor ρ_1, which will not sensibly affect g),

$$\cos\beta\sin\gamma\,d\beta+\sin\beta\cos\gamma\,d\gamma=-l\cos Q\,dQ$$
$$\cos\beta\cos\gamma\,d\beta-\sin\beta\sin\gamma\,d\gamma=\quad l\sin Q\,dQ$$

whence, by eliminating $d\gamma$,

$$\frac{d\beta}{dQ}=\frac{l\sin(Q-\gamma)}{\cos\beta}\qquad .$$

By (505) a sufficiently exact value of ζ_1 for our present purpose is

$$\zeta_1=\cos\beta$$

whence

$$\frac{d\zeta_1}{dQ}=-\sin\beta\,\frac{d\beta}{dQ}$$

$$g=l\sin\beta\sec^2\beta\sin(Q-\gamma)\qquad(532)$$

Putting, finally,

$$G=\cot(Q-E)-\cot(Q-F)=\frac{\sin(E-F)}{\sin(Q-E)\sin(Q-F)}\quad(533)$$

we have

$$dQ = \frac{[5.67664]_x}{G + g} \tag{534}$$

in which 5.67664 is the logarithm of $A \times 206265''$.

When the true value of Q has thus been found, the corresponding latitude and longitude on the earth's surface are found as in Art. 298.

312. The preceding solution of this problem (which is commonly regarded as one of the most intricate problems in the theory of eclipses) is very precise, and the successive approximations converge rapidly to the final result. For practical purposes, however, an extremely precise determination of the limiting curves of the penumbra is of little importance, since no valuable observations are made near these limits. I shall, therefore, now show how the process may be abridged without making any important sacrifice of accuracy. In the first place, it is to be observed that great precision in the angle Q is unnecessary. If LM, Fig. 43, is the limiting curve which is tangent at A to the shadow whose axis is at C, and if Q is in error by the quantity ACA', the point determined will be (nearly) A' instead of A. Now, although A' may be at some distance from A, it is evident that it will still be at a proportionally small distance from the limiting curve. In fact, we may admit an error of several minutes in the value of Q without *sensibly* removing the computed point from the curve. The equation (530), which determines Q, may, therefore, without practical error be written under the approximate form
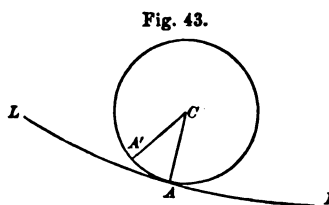
$$e \sin (Q - E) = \zeta_1 f \sin Q$$

and in this we may employ for ζ_1 the value

$$\zeta_1 = \cos \beta$$

Hence, having found β from (502) by employing the first assumed value of Q, we then have

$$\frac{\sin (Q - E)}{\sin Q} = \frac{f \cos \beta}{e}$$

To find the latitude and longitude of the extreme points, we take $\gamma = N + \psi$, $\tan \gamma' = \rho_1 \tan \gamma$, and proceed by (519).

EXAMPLE.—To find the southern limit of the eclipse of July 18, 1860.

First. To find the extreme times.—Taking $T_0 = 2^h$, we have, from our tables, pp. 454, 455, and pp. 464, 465,

$$x_0 = -0.0812 \qquad x' = +0.5452$$
$$y_0 = +0.5961 \qquad y' = -0.1610$$
$$l = 0.5367$$
$$E_0 = 14^\circ\ 17' \qquad b'' = +0.0514$$
$$\log e = 9.7977 \qquad c'' = -0.0151$$

where we take mean values of x', y', &c. From these we find by (537), taking the upper signs in the formulæ,

$$\log m = 9.3554 \qquad\qquad M = 289^\circ\ 32'$$
$$\log n = 9.7182 \qquad\qquad N = 106\ \ 28$$
$$M - N = 183\ \ \ .4$$

Hence, by (538),

$$\log \sin (M - N) = n8.7283 \qquad \log \cos (M - N) = n9.9994$$
$$\log \sin \psi = n8.0837 \qquad \frac{m \cos (M - N)}{n} = +0^h.433$$
$$\log \cos \psi = 0.0000$$
$$\frac{\cos \psi}{n} = \mp 1.913$$
$$\tau = -1.480$$
$$\text{or } \tau = +2.346$$

Therefore, for the first and last points of the curve we have, respectively, the times

$$T_1 = 2^h - 1^h.480 = 0^h.520$$
$$T_2 = 2 + 2.346 = 4.346$$

To find the latitude and longitude of the extreme points corresponding to these times, we have

		First Point.	Last Point.
	ψ	180° 42'	— 0° 42'
	$\gamma = N + \psi$	287 10	105 42
	$\log \tan \gamma$	n0.5102	n0.5492
$\log \rho_1 = 9.9987$	$\log \tan \gamma'$	n0.5089	n0.5479
	d_1	21° 1'.4	20° 59'.8
	μ_1	6 19 .2	63 42.7

Hence, by (519),

ω	102° 40′	339° 30′	
φ	16 5	— 14 47	

Second. To find a series of points on the curve.—We begin by computing the limits of Q for the hours 0ʰ, 1ʰ, 2ʰ, 3ʰ, 4ʰ, 5ʰ. Thus, for 0ʰ we have, from the table p. 465, and by (531),

T	0ʰ
$\log f$	9.3882
$\log e$	9.8019
$\log \tan \nu$	9.5863
ν	21° 5′.6
$\frac{1}{2}E$	2 16.7
$\log \tan (45° + \nu)$	0.3533
$\log \tan \frac{1}{2} E$	8.5997
$\log \tan \psi$	8.9530
ψ	5° 7′.7
$\frac{1}{2}E + \psi$	7 24 .4

For the southern limiting curve Q falls between E and $\frac{1}{2}E + \psi$, *i.e.*, for 0ʰ, between 4° 33′ and 7° 24′. In the same manner we form the other numbers of the following table:

T	Lower limit of Q.	Upper limit of Q.
0ʰ	4° 33′	7° 24′
1	9 22	15 18
2	14 17	24 13
3	19 14	30 53
4	24 8	38 4
5	28 55	44 36

The points of the curve are to be computed for times between 0ʰ.520 and 4ʰ.346, and we shall, therefore, assume for T the series 0ʰ.6, 0ʰ.8, 1ʰ.0 . . . . . 4ʰ.0, 4ʰ.2, which, with the extreme points above computed, will embrace the whole curve.

Instead of determining Q for each of these times by the method of Art. 312, it will be sufficient to determine it for the hours 1ʰ, 2ʰ, 3ʰ, 4ʰ, and, hence, to infer its values for the intervening times. Thus, for $T = 1^h$, assuming $Q = 12°$, which is a

mean between its two limiting values, we proceed by the equations (502), for which we can here use

$$\sin \beta \sin \gamma = x - l \sin Q$$
$$\sin \beta \cos \gamma = y - l \cos Q$$

as follows:

For $T_0 = 1^h$.

x	− 0.6266	
y	+ 0.9170	
l	0.5368	
Assume Q	12°	
$a = x - l \sin Q$	− 0.7382	
$b = y - l \cos Q$	+ 0.3920	
$\log a = \log \sin \beta \sin \gamma$	n9.8682	
$\log b = \log \sin \beta \cos \gamma$	9.5933	
$\log \sin \beta$	9.9221	

$\log \cos \beta$	9.7396
$\log \dfrac{f}{e}$	0.5923
$\log \tan \nu'$	9.3319
ν'	12° 7'.1
$\frac{1}{2} E$	4 41.2
$\log \tan(45° + \nu')$	0.1894
$\tan \frac{1}{2} E$	8.9137
$\tan(Q - \frac{1}{2} E)$	9.1031
$Q - \frac{1}{2} E$	7° 13'.5
Q	11 54.7

We thus find,

for $T =$	1^h	2^h	3^h	4^h
$Q =$	11° 55',	22° 20',	30° 16',	32° 17'.

From these numbers we obtain by simple interpolation sufficiently exact values of Q for our whole series of points. And since it is plain from Art. 312, that even an error of half a degree in Q will not remove the computed point from the true curve by any important amount, we may be content to employ the following series of values as final:

T	Q		T	Q		T	Q		T	Q
$0^h.6$	8°		$1^h.6$	18°		$2^h.6$	28°		$3^h.6$	31°
0 .8	10		1 .8	20		2 .8	29		3 .8	32
1 .0	12		2 .0	22		3 .0	30		4 .0	32
1 .2	14		2 .2	24		3 .2	30		4 .2	32 .5
1 .4	16		2 .4	26		3 .4	31			

For each time T we now take x, y, and l, from the tables of the eclipse, and, with the value of Q for the same time, determine the required point on the outline of the shadow by the

special cases somewhat greater precision might be desired than has been observed in the preceding example. In any such case, recourse may be had to the rigorous method of Art. 311. Since the limits of total or annular eclipse often include but a very narrow belt of the earth's surface, extending nearly equal distances north and south of the curve of central eclipse, they may be derived, with sufficient accuracy for most purposes, from this curve, by a method which will be given in Art. 320.

The curve upon which any given degree of obscuration can be observed may also be computed by the preceding method. It is only necessary to substitute Δ for l, and to give Δ a value corresponding to D according to the equation (529). All the curves thus found begin and end upon the curve of maximum in the horizon.

Curve of Central Eclipse.

315. *To find the curve of central eclipse upon the surface of the earth.*—This curve contains all those points of the surface of the earth through which the axis of the cone of shadow passes. The problem becomes the same as that of Art. 298 upon the supposition that the shadow is reduced to a point—that is, when $l - i\zeta = 0$, and, consequently, by (493),

$$\xi = x \qquad \eta = y$$

Hence, putting

$$y_1 = \frac{y}{\rho_1}$$

the equations (502) to (508) are reduced to the following extremely simple ones, which are rigorously exact:

$$\left.\begin{array}{r}
\sin \beta \sin \gamma = x \\
\sin \beta \cos \gamma = y_1 \\
c \sin C = y_1 \\
c \cos C = \cos \beta \\
\cos \varphi_1 \sin \vartheta = x \\
\cos \varphi_1 \cos \vartheta = c \cos (C + d_1) \\
\sin \varphi_1 = c \sin (C + d_1) \\
\tan \varphi = \dfrac{\tan \varphi_1}{\sqrt{(1 - ee)}} \qquad \omega = \mu_1 - \vartheta
\end{array}\right\} \quad (539)$$

It will be convenient to prepare the values of y_1 for the principal hours of the eclipse; and then for any given time T taking the values of x, y_1, d_1, μ_1, from the eclipse tables, these equations determine a point of the curve.

816. The extreme times between which the solution is possible, or the beginning and end of central eclipse upon the earth, are found as follows. At these instants the axis of the shadow is tangent to the earth's surface, and the central eclipse is observed at sunrise and sunset respectively. Hence, Z being the zenith distance of *the point Z*, we have $\cos Z = 0$, or, by (503), $\zeta_1 = 0$, whence, by (499),

$$\xi^2 + \eta_1^2 = 1$$

or

$$x^2 + y_1^2 = 1$$

which is equivalent to putting $\sin \beta = 1$, or $\cos \beta = 0$, in the first two equations of (539), so that we have

$$\sin \gamma = x, \qquad \cos \gamma = y_1$$

Let x' and y_1' denote the mean hourly changes of x and y_1 computed by the method of Art. 296. Let the required time of beginning or ending be denoted by $T = T_0 + \tau$, T_0 being an arbitrarily assumed epoch; then, if (x) and (y_1) are the values of x and y_1 taken for the time T_0, we have for the time T,

$$\sin \gamma = (x) + x'\tau$$
$$\cos \gamma = (y_1) + y_1'\tau$$

Let m, M, n, N, be determined by the equations

$$m \sin M = (x) \qquad\qquad n \sin N = x'$$
$$m \cos M = (y_1) \qquad\qquad n \cos N = y_1' \qquad\left.\right\}(540)$$

then, from the equations

$$\sin \gamma = m \sin M + n \sin N . \tau$$
$$\cos \gamma = m \cos M + n \cos N . \tau$$

we deduce, in the usual manner,

$$\sin (\gamma - N) = m \sin (M - N)$$
$$\cos (\gamma - N) = m \cos (M - N) + n\tau$$

or, putting $\psi = \gamma - N$, the solution is

$$\sin \psi = m \sin (M - N)$$
$$\tau = \frac{\cos \psi}{n} - \frac{m \cos (M - N)}{n} \qquad\left.\right\}(541)$$
$$T = T_0 + \tau$$

where $\cos \psi$ is to be taken with the negative sign for the beginning and with the positive sign for the end.

To find the latitude and longitude of the extreme points corresponding to these times, we have, in (539), $\cos \beta = 0$, $\sin \beta = 1$, and, therefore, $C = 90°$, $e = \cos \gamma$: hence, taking $\gamma = N + \psi$,

$$\left.\begin{array}{rl}
\cos \varphi_1 \sin \vartheta = & \sin \gamma \\
\cos \varphi_1 \cos \vartheta = & -\cos \gamma \sin d_1 \\
\sin \varphi_1 = & \cos \gamma \cos d_1 \\
\tan \varphi = \dfrac{\tan \varphi_1}{\sqrt{(1 - ee)}} & \qquad \omega = \mu_1 - \vartheta
\end{array}\right\} \quad (542)$$

317. *To find the duration of total or annular eclipse at any point of the curve of central eclipse.*—This is readily obtained from numbers which occur in the previous computations. Let $T =$ the time of central eclipse, $t =$ the duration of total or annular eclipse, then $T' = T \mp \frac{1}{2} t$ is the time of beginning or end. Let x and y be the moon's co-ordinates for the time T; ξ and η those of the point on the earth at this time; x', y', ξ', η', the hourly increments of these quantities; then, at the time T' we have, by (491),

$$(l - i\zeta) \sin Q = x \mp \tfrac{1}{2} x't - (\xi \mp \tfrac{1}{2} \xi't)$$
$$(l - i\zeta) \cos Q = y \mp \tfrac{1}{2} y't - (\eta \mp \tfrac{1}{2} \eta't)$$

But we here have $x = \xi$, $y = \eta$, and we may put $\zeta = \zeta_1 = \cos \beta$, whence

$$(l - i \cos \beta) \sin Q = \mp (x' - \xi') \frac{t}{2}$$

$$(l - i \cos \beta) \cos Q = \mp (y' - \eta') \frac{t}{2}$$

For the values of ξ' and η' we have, with sufficient precision, since t is very small,

$$\xi' = \mu' (- y \cos d + \cos \beta \sin d)$$
$$\eta' = \mu' x \sin d$$

Hence, by (511) and (512), we find, very nearly,

$$x' - \xi' = c' - \mu' \cos d \cos \beta = c' - f \cos \beta$$
$$y' - \eta' = - b'$$

If, therefore, we put

$$L = l - i \cos \beta \qquad\qquad a = c' - f \cos \beta \qquad (543)$$

we have

$$L \sin Q = \frac{at}{2} \qquad\qquad L \cos Q = \frac{b't}{2}$$

where we omit the double sign, since it is only the numerical value of t that is required. Hence, we have, for finding t, the equations

$$\tan Q = \frac{a}{b'} \qquad\qquad t = \frac{7200 \, L \sin Q}{a} \qquad (544)$$

the last equation being multiplied by 3600, so that it now gives t in seconds.

The value of $\cos \beta$ is to be taken from the computation of the central curve for the given time T, and l, $\log i$, $\log f$, c', b', from our eclipse tables.

318. *To find where the central eclipse occurs at noon.*—In this case we have, evidently, $x = 0$, and hence, in (539),

$$\sin \beta = y_1 \qquad\qquad (545)$$

by which β is to be found from the value of y_1 which corresponds to the time when $x = 0$. We then have $C = \beta$, $c = 1$, $\vartheta = 0$, and therefore the required point is found by the formulæ

$$\varphi_1 = \beta + d_1 \qquad\qquad \omega = \mu_1 \qquad (546)$$

in which d_1 and μ_1 are taken for the time when $x = 0$.

319. The formulæ (539), (545), and (546) are not only extremely simple, but also entirely rigorous, and have this advantage over the methods commonly given, that they require no repetition to take into account the true figure of the earth. It may be observed here that the accurate computation of the central curve is of far greater practical importance than that of the limiting curves before treated of.

The formulæ (541) must be computed twice if we wish to obtain the times of beginning and end with the greatest possible precision; for, these times being unknown, we shall have at first to employ the values of x' and y' for the middle of the eclipse, and then to take their values for the times obtained by the first computation of the formulæ. With these new values a second computation will give the exact times.

EXAMPLE.—To compute the curve of central and total eclipse in the eclipse of July 18, 1860.

It is convenient first to prepare the values of $y_1 = \dfrac{y}{\rho_1}$ for the principal hours of the eclipse, as well as its mean hourly differences. With the value log $\rho_1 = 9.99873$ we form, from the values of y given in the table p. 454, the following table:

Gr. T.	y_1	y_1'
0ʰ	+ 0.91972	− 0.16095
1	.75896	114
2	.59782	132
3	.43633	149
4	.27450	166
5	.11237	182

To find the times of beginning and end we may assume $T_0 = 2^h$; and for this time we have

$(x) = m \sin M = -0.08124$ $x' = n \sin N = +0.5453$

$(y_1) = m \cos M = +0.59782$ $y_1' = n \cos N = -0.1613$

whence   log $m =$ 9.78054   log $n =$ 9.7548

$M =$ 352° 15′ 40″ $N =$ 106° 28′.7

Employing but four decimal places in the logarithms for a first approximation, we find, by (541),

$$-\frac{m \cos (M - N)}{n} = + 0^h.435$$

$$\frac{\cos \psi}{n} = \mp 1.468$$

$$\tau_1 = -1.033$$

$$\tau_2 = +1.903$$

Approximate time of beginning $= 2^h - 1^h.033 = 0^h.967$

  "   "   end $= 2 + 1.903 = 3.903$

Taking now x' and y_1' for these times respectively, and repeating the computation, we have

	$C + d_1$	57° 52′ 10″
$\log x = \log \cos \varphi_1 \sin \vartheta$		$n8.90977$
$\log c \cos (C + d_1) = \log \cos \varphi_1 \cos \vartheta$		9.72435
$\log c \sin (C + d_1) = \log \sin \varphi_1$		9.92636
	ϑ	351° 17′ 13″
	μ_1	28 31 12
	ω	37 13 59
	φ	57 39 20
App. Time $= \vartheta$ in time,		$23^h 25^m 8^s.8$

For the duration of totality at this point, we take from pp. 454, 464, 465,

$$l = - 0.009082 \qquad\qquad b' = + 0.1532$$
$$\log i = \quad 7.6608 \qquad\qquad c' = + 0.6011$$
$$\log f = \quad 9.3883$$

and hence, with $\log \cos \beta = 9.9017$ above found, we obtain, by (543),

$$L = - 0.012734 \qquad\qquad a = + 0.4061$$

and, by (544), disregarding the negative sign of L,

$$t = 211^s.3 = 3^m 31^s.3$$

For the place where the central eclipse occurs at noon, we find that $x = 0$ at the time $T = 2^h.149$, at which time we have

$y_1 = \sin \beta$		$+ 0.57378$
	β	35° 0′ 53″
	d_1	21 0 45
	φ_1	56 1 38
	φ	56 6 57
$\mu_1 = \omega$		30 45 18

The whole curve may be traced through the points given in the following table:

SOLAR ECLIPSE, July 18, 1860.—CURVE OF CENTRAL AND TOTAL ECLIPSE.

Greenwich Mean Time.	Latitude. ϕ		Long. W. from Greenwich. ω		App. Local Time. ϑ		Duration of Totality.	
0ʰ.967	45°	36'.4	126°	3'.1	16ʰ	27ᵐ.9		
1 .0	50	37 .8	113	11 .6	17	21 .3	2ᵐ	1ˢ.5
1 .2	57	16 .2	89	14 .6	19	9 .1	2	35 .1
1 .4	59	29 .1	72	52 .8	20	26 .6	2	55 .8
1 .6	59	55 .1	59	5 .2	21	33 .7	3	11 .4
1 .8	59	11 .6	47	16 .6	22	33 .0	3	23 .1
2 .0	57	39 .3	37	14 .0	23	25 .1	3	31 .3
2 .149	56	7 .0	30	45 .3	0	0 .0	3	34 .7
2 .2	55	31 .5	28	42 .6	0	11 .2	3	36 .2
2 .4	52	56 .9	21	25 .1	0	52 .4	3	38 .0
2 .6	50	0 .9	15	8 .9	1	29 .8	3	36 .4
2 .8	46	46 .3	9	21 .8	2	4 .6	3	32 .0
3 .0	43	13 .6	4	2 .2	2	37 .9	3	24 .6
3 .2	39	20 .7	358	47 .1	3	10 .9	3	14 .4
3 .4	35	1 .6	353	12 .5	3	45 .3	3	1 .1
3 .6	30	1 .5	346	35 .4	4	23 .7	2	43 .5
3 .8	23	28 .5	336	44 .1	5	15 .1	2	18 .5
3 .904	15	45 .6	320	53 .2	6	24 .8		

Northern and Southern Limits of Total or Annular Eclipse.

320. *To find the northern and southern limits of total or annular eclipse.*—As already remarked in Art. 314, these limits may be rigorously determined by the method of Art. 311, by taking $l =$ the radius of the umbra (*i.e.* for interior contacts); but I here propose to deduce them from the previously computed curve of central eclipse. This radius l is assumed to be so small that we may neglect its square, which can seldom exceed .0003, and this degree of approximation will in the greater number of cases suffice to determine points on the limits within 2' or 3', which is practically quite accurate enough.

The two limiting curves of total or annular eclipse, then, lie so near to the central curve that the value $\zeta_1 = \cos \beta$, for a given time T, already found in the computation of the latter curve, may be used for the former in the approximate equation which determines Q. We can, therefore, immediately find Q by (535),—*i.e.*

These values are yet to be divided by sin 1′ to reduce them to minutes of arc. It will be convenient to put

$$
\left.
\begin{aligned}
l' &= \frac{l}{\sin 1'} \qquad\qquad\qquad i' = \frac{i}{\sin 1'} \\[2ex]
\lambda &= \frac{l - i \cos \beta}{\cos \beta \sin 1'} = \frac{l'}{\cos \beta} - i'
\end{aligned}
\right\} \quad (548)
$$

in which l', i', and λ will be expressed in minutes.

We may in practice substitute $d\varphi$ for $d\varphi_1$, within the limits of accuracy we have adopted; for we find, from the equations on p. 457,

$$
d\varphi = \frac{d\varphi_1}{\sqrt{(1-ee)}} \cdot \frac{\cos^2 \varphi}{\cos^2 \varphi_1} = d\varphi_1 \frac{1 - ee \sin^2 \varphi}{\sqrt{(1 - ee)}}
$$

where the multiplier of $d\varphi_1$ cannot differ more from unity than $\sqrt{(1 - ee)}$ does,—i.e. not more than 0.00335: so that the substitution of one for the other will never produce an error of 1′ so long as $d\varphi_1$ is less than 5°.

Finally, adapting the values of $d\omega$ and $d\varphi$ for logarithmic computation, by putting

$$
\left.
\begin{aligned}
h \sin H &= \cos Q \\
h \cos H &= \sin Q \sin d_1
\end{aligned}
\right\} \quad (549)
$$

we have

$$
\begin{aligned}
d\omega &= \lambda\, [h \cos (\vartheta - H) \tan \varphi_1 + \sin Q \cos d_1] \\
d\varphi &= \lambda\, h \sin (\vartheta - H)
\end{aligned}
$$

The formulæ (547) give two values of Q differing 180°. The second value will evidently give the same numerical values of $d\omega$ and $d\varphi$, but with opposite signs; and therefore we may compute the equations (549) with only the acute value of Q, and then the longitude and latitude of a point on one of the limits are

$$
\omega + d\omega, \qquad\qquad \varphi + d\varphi
$$

and those of a point on the other limit are

$$
\omega - d\omega, \qquad\qquad \varphi - d\varphi
$$

The first of these limits will be the northern in the case of total eclipse, but the southern in the case of annular eclipse, observing always to take l with the negative sign for total eclipse, as it comes out by the formulæ (487) and (489).

It is evident that this approximate method is not accurate when $\cos \beta$ is very small, that is, near the extreme points of the curves; and it fails wholly for these points themselves, since $\cos \beta$ is then zero and the value of λ becomes infinite. These extreme points, however, are determined directly in a very simple manner by the formulæ (536), (537), (538), combined with (519), by employing in (536) and (537) the value of l for interior contacts; and it is with these formulæ, therefore, that the computation of the limits of total or annular eclipse should be commenced.

EXAMPLE.—Find the northern and southern limits of total eclipse in the eclipse of July 18, 1860.

First. To find the extreme points.—The values of b' and c' for exterior contacts, from which the values of E on p. 465 are derived, differ so little from those for interior contacts that in practice, unless extreme precision is required, we may dispense with the computation of the latter. For our present example, therefore, taking the value of E for $T_0 = 2^h$ and the mean value of $\log e$, as in the computation of the extreme points of the southern limit for the penumbra, p. 487, together with

$$l = -0.0091$$

we find, by (536) and (537), for the northern limit,

$$\log m = 9.7854 \qquad M = 352° 33'.6$$
$$\log n = 9.7553 \qquad N = 106 \quad 27'.0$$

and for the southern limit,

$$\log m = 9.7731 \qquad M = 351° 55'.0$$
$$\log n = 9.7542 \qquad N = 106 \quad 27.0$$

Hence, by (538),

	Northern Limit.		Southern Limit.	
	First Point.	Last Point.	First Point.	Last Point.
ψ	213° 54'.3	326° 5'.7	212° 39'.0	327° 21'.0
T	$0^h.976$	$3^h.892$	$0^h.951$	$3^h.917$

Taking $\gamma = N + \psi$, and the values of d_1 and μ_1 for these times respectively, with $\log \rho_1 = 9.9987$, we find, by (518) and (519),

γ	320° 21'.3	72° 32'.7	319° 6'.0	73° 48.0
log tan γ'	n9.9170	0.5012	n9.9363	0.5355
d_1	21° 1'.2	21° 0'.0	21° 1'.2	21° 0'.0
ϑ	246 31.7	96 26.7	247 26.7	95 57.7
μ_1	13 9.6	56 54.1	12 47.1	57 16.6
ω	126 37.9	320 27.4	125 20.4	321 18.9
φ	46 7.7	16 21.6	45 2.8	15 11.4

Second. To find a series of points between these extremes, by the aid of the curve of central eclipse, we assume the same series of times as in the computation of that curve, and proceed by (547), (548), and (549); to illustrate the use of which I add the computation for $T = 2^h$ in full. From the computation, p. 496, we have

For $T = 2^h$		
	log cos β	9.9017
	log tan φ_1	0.1970
	ϑ	351° 17'.2
	d_1	21 0.8
	ω	37 14.0
	φ	57 39.3

Then, by (547),

(p. 465) log $\dfrac{f}{e}$	9.5953
log cos β	9.9017
log tan ν'	9.4970
ν'	17° 26'.0
$\frac{1}{2}E$	7 8.7
log tan $(45° + \nu')$	0.2823
log tan $\frac{1}{2}E$	9.0982
log tan $(Q - \frac{1}{2}E)$	9.3805
$Q - \frac{1}{2}E$	13° 30'.3
Q	20 39.0

By (548),

l	— 0.009082
log l	n7.9582
log l'	n1.9945
log i	7.6608
log i'	1.1971
i'	15'.74
l' sec β	— 39.14
λ	— 54.88

Hence, by (549),

log cos Q = log h sin H	9.9712
log sin Q sin d_1 = log h cos H	9.1020
log h	9.9751
H	82° 18'.2
$\vartheta - H$	268 59.0

latter being negative) for the place of observation. From (488) we find, by putting sec $f = 1$,

$$L + L_1 = 2(L - k)$$

and hence

$$D = \frac{L - \Delta}{2(L - k)} \tag{558}$$

in which $k = 0.2723$.

If we neglect the augmentation of the moon's diameter, or, which is equivalent, the small difference between L and l, and put

$$\left. \begin{aligned} \varepsilon &= \frac{l}{2(l - k)} \\ D &= \varepsilon \mp \varepsilon \sin \psi \end{aligned} \right\} \tag{559}$$

we have

where the lower sign is to be used when $\sin \psi$ is negative, so that D is always the *numerical* difference of ε and $\varepsilon \sin \psi$. In this form ε may be computed for the eclipse generally, and ψ will be derived from the computation for the penumbra for the given place. A preference should be given to the value of ψ found from the computation for the time nearest to that of greatest obscuration, which is usually that used in the first approximation of Art. 322.

EXAMPLE.—Find the time of beginning and end, &c., of the eclipse of July 18, 1860, at Cambridge, Mass.

The latitude and longitude are

$$\varphi = 42° 22' 49'' \qquad \omega = 71° 7' 25''$$

For this latitude we find, by the aid of Table III., or by the formulæ (87),

$$\log \rho \sin \varphi' = 9.82644 \qquad \log \rho \cos \varphi' = 9.86912$$

With the aid of the chart, p. 504, we estimate the time of the middle of the eclipse at Cambridge to be not far from 1^h. Hence, taking $T_0 = 1^h$ for our first approximation, we take for this time, from the eclipse tables, p. 454,

$x = -0.6266$	$x' = +0.5453$	$l = 0.5868$
$y = +0.7567$	$y' = -0.1605$	$\log i = 7.66287$
$d = 20° 57'.4$	$\mu_1 = 13° 31'.2$	$\log \mu' = 9.41799$

Hence, by (550) and (551),

$$\mu_1 - \omega = \vartheta = \quad 302°\ 23'.8$$
$$B = \quad 59\ \ 24.6$$
$$\xi = -\ 0.6246$$
$$\eta = +\ 0.4844$$
$$\xi' = +\ 0.1038$$
$$\eta' = -\ 0.0585$$

$$\log \zeta = \quad 9.7853$$
$$i\zeta = \quad 0.0028$$
$$L = l - i\zeta = \quad 0.5340$$

and, by (552) and (553),

$$m \sin M = x - \xi = -\ 0.0020$$
$$m \cos M = y - \eta = +\ 0.2723$$
$$\log m = \quad 9.4350$$
$$M = \quad 359°\ 34'.7$$
$$M - N = \quad 256\ \ 34.1$$
$$\log \sin \psi = \quad n9.6955$$
$$\log \cos \psi = \quad 9.9887$$

$$n \sin N = x' - \xi' = +\ 0.4415$$
$$n \cos N = y' - \eta' = -\ 0.1020$$
$$\log n = \quad 9.6562$$
$$N = \quad 103°\ 0'.6$$
$$-\frac{m \cos (M - N)}{n} = +\ 0^h.140$$
$$\frac{L \cos \psi}{n} = \mp\ 1.023$$
$$\tau = \begin{cases} -\ 0.883 \\ \text{or} +\ 1.163 \end{cases}$$

Approximate time of beginning $= 0^h.117$
         "   "   end $= 2.163$

Taking then for a second approximation $T_0 = 0^h.12$ for beginning, and $T_0 = 2^h.16$ for end, we shall find*

	Beginning.	End.
T_0	$0^h.12$	$2^h.16$
x	$-\ 1.10642$	$+\ 0.00601$
y	$+\ 0.89783$	$+\ 0.57034$
x'	$+\ 0.54528$	$+\ 0.54530$
y'	$-\ 0.16015$	$-\ 0.16090$
d	$20°\ 57'\ 45''$	$20°\ 56'\ 53''$
μ_1	$0\ \ 19\ \ 8$	$30\ \ 55\ \ 13$
l	0.53686	0.53673
ϑ	$289°\ 11'\ 43''$	$319°\ 47'\ 48''$
ξ	$-\ 0.69868$	$-\ 0.47755$
η	$+\ 0.53915$	$+\ 0.42423$
$\log \zeta$	9.66935	9.88504
ξ'	$+\ 0.06368$	$+\ 0.14793$
η'	$-\ 0.06544$	$-\ 0.04470$

---

* The values of x' and y' here employed are not those given in the table p. 445, but their actual values for the time T_0, as given in the table of x' and y' on p. 464.

	Beginning.	End.
$i\zeta$	0.00215	0.00353
L	0.53471	0.53320
$m \sin M$	-0.40774	$+0.48356$
$m \cos M$	$+0.35868$	$+0.14611$
$\log m$	9.73484	9.70342
M	311° 20′ 16″	73° 11′ 15″
$n \sin N$	$+0.48160$	$+0.39737$
$n \cos N$	-0.09471	-0.11620
$\log n$	9.69093	9.61702
N	101° 7′ 32″	106° 18′ 0″
$M-N$	210 12 44	326 53 15
ψ	210 44 0	328 49 56
$M-N-\psi$	$-$ 31′ 16″	$-$ 1° 56′ 41″
τ	$+0^h.0197$	$+0^h.0800$
$T\Big\{$	$0^h.1397$	2 .2400
	$0^h\ 8^m\ 23^s$	$2^h\ 14^m\ 24^s$
ω	4 44 30	4 44 30
Local time, $\quad t\Big\{$	19 23 53	21 29 54
	July 17.	July 17.
Angle of Pt.of Contact from North Pt. of the sun = $\Big\}$ $Q = N + \psi$	311° 51′ 32″	75° 7′ 56″

A third approximation, commencing with the last computed times, changes them by only a fraction of a second.

To find the angular distance of the point of contact from the *vertex* of the sun's limb, we have from the second approximation, by (554) and (555),

	Beginning.	End.
$\xi + \xi'\tau = p \sin \gamma$	-0.6974	-0.4658
$\eta + \eta'\tau = p \cos \gamma$	$+0.5379$	$+0.4206$
γ	307° 38′.8	312° 4′.5
Angle from vertex $= Q - \gamma$	4 12.7	123 3 .4

The time of greatest obscuration is best found from the first approximation, which gives, by (556),

in which μ' denotes the change of μ_1 in one second, and is the same as the μ' of our former method divided by 3600.

To find m' we have, following the same notation, and neglecting the small changes of E, F, G, H, l, and f,

$$B' = \quad y' = C'$$
$$b' = \quad B' - \mu' G \rho \cos \varphi' \sin \vartheta$$
$$c' = - C' + \mu' H \rho \cos \varphi' \sin \vartheta$$

Since f is small, we may in these approximate expressions put $G = H$, and hence

$$b' = - c' = B' - \mu' G \rho \cos \varphi' \sin \vartheta \qquad (561^*)$$

Now, from the formula $m^2 = bc$, we derive

$$2 mm' = cb' + bc' = (c - b) b'$$

$$m' = \tfrac{1}{2} \left(\sqrt{\tfrac{c}{b}} - \sqrt{\tfrac{b}{c}} \right) b'$$

which, if we assume

$$\tan \tfrac{1}{2} Q = \sqrt{\tfrac{c}{b}} = \frac{c}{m} = \frac{m}{b} \qquad (562)$$

becomes

$$m' = - b' \cot Q$$

and therefore τ is found by the formula

$$\tau = \frac{m - a}{a' + b' \cot Q} \qquad (563)$$

The Ephemeris gives also the values of A', B', and C', which are the changes of A, B, and C in one second. These changes being very small, the unit adopted in expressing them is .000001; so that the above value of τ, as also the value of μ' in (561), must be multiplied by 10^6. The formulæ (560–563) then agree with those given in the explanation appended to the Ephemeris.

It is easily seen that Q here denotes the same angle as in the preceding articles; for we have at the instant of contact

$$\tan Q = - \frac{b'}{m'} = \frac{2 m}{b - c} = \frac{x - \xi}{y - \eta}$$

Examples of the application of this method are given in every volume of the American Ephemeris.

326. The preceding articles embrace all that is important in relation to the prediction of solar eclipses. Since absolute rigor is not required in mere predictions, I have thus far said nothing of the effect of refraction, which, though extremely small, must be treated of before we proceed to the application of *observed* eclipses, where the greatest possible degree of precision is to be sought.

CORRECTION FOR ATMOSPHERIC REFRACTION IN ECLIPSES.

327. That the refraction varies for bodies at different distances from the earth has already been noticed in Art. 106; but the difference is so small that it is disregarded in all problems in which the absolute position of a single body is considered. Here, however, where two points at very different distances from the earth are observed in apparent contact, it is worth while to inquire how far the difference in question may affect our results.

Let *SMDA*, Fig. 44, be the path of the ray of light from the sun's limb to the observer at *A*, which touches the moon's limb at *M; SMB* the straight line which coincides with this path between *S* and *M*, but when produced intersects the vertical line of the observer in *B*. It is evident that the observer at *A* sees an apparent contact of the limbs at the instant when an observer at *B* would see a true contact if there were no refraction. Hence, if we substitute the point *B* for the point *A* in the formula of the eclipse, we shall fully take into account the effect of refraction.

Fig. 44.
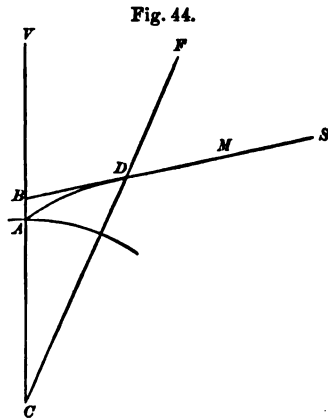
For the purpose of determining the position of the point *B*, whose distance from *A* is very small, it will suffice to regard the earth as a sphere with the radius $\rho = CA$. It is one of the properties of the path of a ray of light in the atmosphere that the product $q\mu \sin i$ is constant (Art. 108), q denoting the normal to any infinitesimal stratum of the atmosphere at the point in which the ray intersects the stratum, μ the index of refraction of that stratum, and i the angle which the ray makes with the normal.

If, then, ρ, μ_0, Z' denote the values of q, μ, and i for the point A, we have, as in the equation (149),

$$q\mu \sin i = \rho\mu_0 \sin Z'$$

in which Z' is the apparent zenith distance of the point M, and μ_0 is the index of refraction of the air at the observer.

Now, let us consider the normal q to be drawn to a point D of the ray where the refractive power of the air is zero, that is, to a point in the rectilinear portion of the path where $\mu = 1$. Then our equation becomes

$$q \sin i = \rho\mu_0 \sin Z'$$

in which $q = CD$, $i = MDF = CDB$. Putting $Z = $ the true zenith distance of $M = MBV$, and $s = $ the height of B above the surface of the earth $= AB$, the triangle CDB gives

$$(\rho + s) \sin Z = q \sin i$$

which with the preceding equation gives

$$1 + \frac{s}{\rho} = \frac{\mu_0 \sin Z'}{\sin Z} \tag{564}$$

In order to substitute the point B for the point A in our computation of an eclipse, we have only to write $\rho + s$ for ρ in the equations (483), or $\rho\left(1 + \dfrac{s}{\rho}\right)$ for ρ. Therefore, when we have computed the values of $\log \xi$, $\log \eta$, and $\log \zeta$ by those equations in their present form, we shall merely have to correct them by adding to each the value of $\log\left(1 + \dfrac{s}{\rho}\right)$. This logarithm may be computed by (564) for a mean value of μ_0 ($= 1.0002800$) and for given values of Z. For Z we may take the true zenith distance of *the point Z* (Art. 289), determined by a and d. But by the last equation of (483) we have so nearly $\cos Z = \zeta$ that in the table computed by (564) we may make $\log \zeta$ the argument, as in the following table, which I have deduced from that of BESSEL (*Astron. Untersuchungen*, Vol. II. p. 240).

log ζ	Correction of logs. of ξ, η, ζ.	log ζ	Correction of logs. of ξ, η, ζ.
0.0	0.0000000	8.5	0.0000446
9.9	.0000001	8.4	.0000525
9.8	.0000002	8.3	.0000602
9.7	.0000005	8.2	.0000672
9.6	.0000008	8.1	.0000734
9.5	0.0000014	8.0	0.0000788
9.4	.0000023	7.9	.0000835
9.3	.0000035	7.8	.0000875
9.2	.0000054	7.7	.0000909
9.1	.0000081	7.6	.0000937
9.0	0.0000119	7.4	0.0000978
8.9	.0000167	7.2	.0001006
8.8	.0000225	7.0	.0001023
8.7	.0000292	6.5	.0001044
8.6	.0000367	6.0	.0001051
8.5	0.0000446	$-\infty$	0.0001054

The numbers in this table correspond to that state of the atmosphere for which the refraction table (Table II.) is computed; that is, for the case in which the factors β and γ of that table are each $= 1$. For any other case the tabular logarithm is to be varied in proportion to β and γ.

It is evident from this table that the effect of refraction will mostly be very small, for so long as the zenith distance of the moon is less than 70° we have log $\zeta > 9.53$, and the tabular correction less than .000001. From the zenith distance 70° to 90° the correction increases rapidly, and should not be neglected.

CORRECTION FOR THE HEIGHT OF THE OBSERVER ABOVE THE LEVEL OF THE SEA.

328. If s' is the height of the observer above the level of the sea, it is only necessary to put $\rho + s'$ for ρ in the general formulæ of the eclipse; and this will be accomplished by adding to log ξ, log η, and log ζ the value of log $\left(1 + \dfrac{s'}{\rho}\right)$, which is ($M$ being the modulus of common logarithms)

$$M\left[\frac{s'}{\rho} - \tfrac{1}{2}\left(\frac{s'}{\rho}\right)^2 + \&c.\right]$$

But s' is always so small in comparison with ρ that we may

neglect all but the first term of this formula; and hence, by taking a mean value of ρ (for latitude 45°) and supposing s' to be expressed in *English feet*, we find

$$\text{Correction of } \log \xi, \log \eta, \log \zeta = 0.00000002079 \, s' \quad (565)$$

For example, if the point of observation is 1000 feet above the level of the sea, we must increase the logarithms of ξ, η, and ζ by 0.0000208.

If s' is expressed in *metres*, the correction becomes 0.000000064 s'.

APPLICATION OF OBSERVED ECLIPSES TO THE DETERMINATION OF TERRESTRIAL LONGITUDES AND THE CORRECTION OF THE ELEMENTS OF THE COMPUTATION.

329. *To find the longitude of a place from the observation of an eclipse of the sun.*—The observation gives simply the local times of the contacts of the disc of the sun and moon: in the case of partial eclipse, two exterior contacts only; in the case of total or annular eclipse, also two interior contacts.

Let

$\omega =$ the west longitude of the place,
$t \;=$ the local mean time of an observed contact,
$\mu =$ the corresponding local sidereal time.

The conversion of t into μ requires an approximate knowledge of the longitude, which we may always suppose the observer to possess, at least with sufficient precision for this purpose.

Let T_0 be the adopted epoch from which the values of x and y are computed (Art. 296), and let

$x_0, y_0 =$ the values of x and y at the time T_0,
$x', y' =$ their *mean* hourly changes for the time $t + \omega$;

then, if we also put

$$\tau = t + \omega - T_0 \quad (566)$$

the values of x and y at the time $t + \omega$ (which is the time at the first meridian when the contact was observed) are

$$x_0 + x'\tau, \qquad\qquad y_0 + y'\tau$$

The values of x' and y' to be employed in these expressions may be taken for the time $t + \omega$ obtained by employing the

approximate value of ω, and will be sufficiently precise unless the longitude is very greatly in error.

The quantities l and i change so slowly that their values taken for the approximate time $t + \omega$ will not differ sensibly from the true ones. For the same reason, the quantities a and d taken for this time will be sufficiently precise: so that, the latitude being given, the co-ordinates ξ, η, ζ of the place of observation may be correctly found by the formulæ (483). Since, then, at the instant of contact the equation (490) or (491) must be exactly satisfied, we have, putting $L = l - i\zeta$,

$$\left. \begin{aligned} L \sin Q &= x_0 - \xi + x'\tau \\ L \cos Q &= y_0 - \eta + y'\tau \end{aligned} \right\} \quad (567)$$

in which τ is the only unknown quantity. Let the auxiliaries m, M, n, N be determined by the equations

$$\left. \begin{aligned} m \sin M &= x_0 - \xi & n \sin N &= x' \\ m \cos M &= y_0 - \eta & n \cos N &= y' \end{aligned} \right\} \quad (568)$$

then, from the equations

$$\begin{aligned} L \sin Q &= m \sin M + n \sin N \cdot \tau \\ L \cos Q &= m \cos M + n \cos N \cdot \tau \end{aligned}$$

by putting $\psi = Q - N$, we obtain

$$\left. \begin{aligned} \sin \psi &= \frac{m \sin (M - N)}{L} \\[2mm] \tau &= \frac{L \cos \psi}{n} - \frac{m \cos (M - N)}{n} \\[2mm] &= \frac{m}{n} \cdot \frac{\sin (M - N - \psi)}{\sin \psi} \end{aligned} \right\} \quad (569)$$

where the second form for τ will be the more convenient except when $\sin \psi$ is very small. As in the similar formulæ (553), the angle ψ must be so taken that $L \cos \psi$ shall be negative for first contacts and positive for last contacts, remembering that in the case of total eclipse L is a negative quantity.

Having found τ, the longitude becomes known by (566), which gives

$$\omega = T_0 - t + \tau \quad\quad (570)$$

If the observed local time is sidereal, let μ_0 be the sidereal time at the first meridian, corresponding to T_0; then, τ being reduced to sidereal seconds, we shall have

$$\omega = \mu_0 - \mu + \tau$$

and this process will be free from the theoretical inaccuracy arising from employing an approximate longitude in converting μ into t.

The unit of τ in (569) is one mean hour; but, if we write

$$\tau = \frac{h\,L\,\cos\psi}{n} - \frac{h\,m\,\cos(M-N)}{n}$$

$$= h \cdot \frac{m}{n} \cdot \frac{\sin(M-N-\psi)}{\sin\psi}$$

we shall find τ in mean or sidereal seconds, according as we take $h = 3600$, or $h = 3609.856$.

330. The rule given in the preceding article for determining the sign of $\cos\psi$ (which is that usually given by writers on this subject) is not without exception in theory, although in practice it will be applicable in all cases where the observations are suitable for finding the longitude with precision; and, were an exceptional case to occur in practice, a knowledge of the approximate longitude would remove all doubt as to the sign of the term $\frac{L\cos\psi}{n}$. But it is is easy to deduce the mathematical condition for this case.

At the instant of contact, the quantity

$$(x_0 - \xi + x'\tau)^2 + (y_0 - \eta + y'\tau)^2$$

is equal to L^2. At the next following instant, when τ becomes $\tau + d\tau$, it is less or greater than L^2 according as the eclipse is beginning or ending. If then we regard L^2 as sensibly constant, the differential coefficient of this quantity relatively to the time must be negative for first and positive for last contacts. The half of this coefficient is

$$(x_0 - \xi + x'\tau)(x' - \xi') + (y_0 - \eta + y'\tau)(y' - \eta')$$

(where the derivatives of ξ and η are denoted by ξ' and η'), or, by (567), putting $N + \psi$ for Q,

$$L\left[\sin(N+\psi)(x'-\xi') + \cos(N+\psi)(y'-\eta')\right]$$

Computing ξ' and η' by the formulæ (551), or, in this case, by

$$\xi' = \mu'\rho \cos \varphi' \cos (\mu - a) \qquad \eta' = \mu'\xi \sin d$$

and putting

$$n' \sin N' = x' - \xi' \qquad n' \cos N' = y' - \eta'$$

the above expression becomes

$$L \, n' \cos (N - N' + \psi)$$

Hence, when L is positive, that is, for exterior contacts and interior contacts in annular eclipse, ψ must be so taken that $\cos (N - N' + \psi)$ shall be negative for first and positive for last contact. That is, *for first contact ψ must be taken between $N' - N + 90°$ and $N' - N + 270°$; and for last contact between $N' - N + 90°$ and $N' - N - 90°$.* For total eclipse, invert these conditions.

In Art. 322, we have $N = N'$, and hence the rule given for the case there considered is always correct.

331. *To investigate the correction of the longitude found from an observed solar eclipse, for errors in the elements of the computation.*
Let

$\Delta x, \Delta y, \Delta L =$ the corrections of x, y, and L, respectively, for errors of the Ephemeris,
$\Delta \xi, \Delta \eta =$ the corrections of ξ and η for errors in ρ and φ',
$\Delta \tau =$ the resulting correction of τ.

The relation between these corrections, supposing them very small, will be obtained by differentiating the values of $L \sin Q$ and $L \cos Q$ of the preceding article, by which we obtain

$$\Delta L \sin Q + L \cos Q \, \Delta Q = \Delta x - \Delta \xi + x' \Delta \tau$$
$$\Delta L \cos Q - L \sin Q \, \Delta Q = \Delta y - \Delta \eta + y' \Delta \tau$$

where Δx and Δy, being taken to denote the corrections of $x = x_0 + x'\tau$ and $y = y_0 + y'\tau$, include the corrections of x' and y'. Substituting in these equations $n \sin N$ for x' and $n \cos N$ for y', and eliminating ΔQ, we find

$$\Delta L = (\Delta x - \Delta \xi) \sin Q + (\Delta y - \Delta \eta) \cos Q + n \cos (Q - N) \cdot \Delta \tau$$

and substituting for Q its value $N + \psi$,

$$\Delta \tau = - (\Delta x - \Delta \xi) \frac{\sin (N + \psi)}{n \cos \psi} - (\Delta y - \Delta \eta) \frac{\cos (N + \psi)}{\cos \psi} + \frac{\Delta L}{n \cos \psi}$$

or

$$\Delta\tau = -\frac{1}{n}(\Delta x \sin N + \Delta y \cos N) + \frac{1}{n}(-\Delta x \cos N + \Delta y \sin N)\tan\psi$$

$$+\frac{1}{n}(\Delta\xi \sin N + \Delta\eta \cos N) - \frac{1}{n}(-\Delta\xi \cos N + \Delta\eta \sin N)\tan\psi$$

$$+\frac{\Delta L \sec\psi}{n} \tag{571}$$

which is at once the correction of τ and of the longitude, since we have, by (570), $\Delta\omega = \Delta\tau$.

332. In this expression for $\Delta\tau$, the corrections Δx, Δy, &c. have particular values belonging to the given instant of observation or to the given place. In order to render it available for determining the corrections of the original elements of computation, we must endeavor to reduce it to a function of quantities which are constant during the whole eclipse and independent of the place of observation. For this purpose, let us first consider those parts of $\Delta\tau$ which involve Δx and Δy. For any time T_1, at the first meridian, we have

$$x = x_0 + n \sin N (T_1 - T_0)$$
$$y = y_0 + n \cos N (T_1 - T_0)$$

whence

$$x \sin N + y \cos N = \quad x_0 \sin N + y_0 \cos N + n(T_1 - T_0)$$
$$- x \cos N + y \sin N = - x_0 \cos N + y_0 \sin N$$

The last of these expressions, being independent of the time, is constant. If we denote it by $\varkappa$; that is, put

$$\varkappa = - x_0 \cos N + y_0 \sin N = - x \cos N + y \sin N \tag{572}$$

we shall find from the two expressions

$$xx + yy = \varkappa\varkappa + [x_0 \sin N + y_0 \cos N + n(T_1 - T_0)]^2 \tag{573}$$

This equation shows that the quantity $\sqrt{xx+yy}$, which is the distance of the axis of the shadow from the centre of the earth, can never be less than the constant $\varkappa$, and it attains this minimum value when the second term vanishes, that is, when

$$x_0 \sin N + y_0 \cos N + n(T_1 - T_0) = 0$$

and hence when

$$T_1 = T_0 - \frac{1}{n}(x_0 \sin N + y_0 \cos N) \tag{574}$$

where we have multiplied by h to reduce to seconds. The unit is either one second of mean or one second of sidereal time, according as τ is in mean or sidereal time. If the former, we take $h = 3600$; if the latter, $h = 3610$.

333. The transformations of the preceding article have led us to an expression in which the corrections ΔT_1, Δx, Δn, and Δee are all constants for the earth generally, and which, therefore, have the same values in all the equations of condition formed from the observations in various places. But a still further transformation is necessary if we wish the equation to express the relation between the longitude and the corrections of the Ephemeris, so that we may finally be enabled not only to correct the longitudes, but also the Ephemeris.

Since ΔT_1, Δx, Δn are constant for the whole eclipse, we can determine them for any assumed time, as the time T_1 itself. For this time we have

$$\left. \begin{array}{l} x \sin N + y \cos N = 0 \\ x \cos N + y \sin N = \varkappa \\ \Delta x \sin N + \Delta y \cos N = - n \Delta T_1 \\ - \Delta x \cos N + \Delta y \sin N = \Delta \varkappa \end{array} \right\} \quad (578)$$

The general values of x and y (482) may be thus expressed:

$$x = \frac{X}{\sin \pi} \qquad y = \frac{Y}{\sin \pi}$$

where

$$X = \cos \delta \sin (\alpha - a) \qquad Y = \sin \delta \cos d - \cos \delta \sin d \cos (\alpha - a)$$

From these we deduce

$$\Delta x = \frac{\Delta X}{\sin \pi} - x \frac{\Delta \pi}{\tan \pi} \qquad \Delta y = \frac{\Delta Y}{\sin \pi} - y \frac{\Delta \pi}{\tan \pi}$$

whence

$$\Delta x \sin N + \Delta y \cos N = \frac{\Delta X \sin N + \Delta Y \cos N}{\sin \pi} - (x \sin N + y \cos N) \frac{\Delta \pi}{\tan \pi}$$

$$- \Delta x \cos N + \Delta y \sin N = \frac{-\Delta X \cos N + \Delta Y \sin N}{\sin \pi} + (x \cos N - y \sin N) \frac{\Delta \pi}{\tan \pi}$$

and for the time T_1 these become, according to (578),

$$- n \Delta T_1 = \frac{\Delta X \sin N + \Delta Y \cos N}{\sin \pi}$$

$$\Delta \varkappa = \frac{-\Delta X \cos N + \Delta Y \sin N}{\sin \pi} - \varkappa \frac{\Delta \pi}{\tan \pi}$$

Again, by differentiating the values of X and Y, we have

$$\Delta X = \cos \delta \cos (\alpha - a) \Delta(\alpha - a) - \sin \delta \sin (\alpha - a) \Delta\delta$$
$$\Delta Y = [\cos \delta \cos d + \sin \delta \sin d \cos (\alpha - a)] \Delta\delta$$
$$\quad - [\sin \delta \sin d + \cos \delta \cos d \cos (\alpha - a)] \Delta d$$
$$\quad + \cos \delta \sin d \sin (\alpha - a) \Delta(\alpha - a)$$

But for the time of nearest approach we may take $\alpha = a$ and put $\cos(\delta - d) = 1$, whence

$$\Delta X = \cos \delta . \Delta(\alpha - a) \qquad\qquad \Delta Y = \Delta(\delta - d)$$

so that

$$\left.\begin{array}{l} -n\Delta T_1 = \dfrac{\sin N \cos \delta . \Delta(\alpha - a) + \cos N . \Delta(\delta - d)}{\sin \pi} \\[3mm] \Delta x = \dfrac{-\cos N . \cos \delta \Delta(\alpha - a) + \sin N . \Delta(\delta - d)}{\sin \pi} - x\dfrac{\Delta\pi}{\tan \pi} \end{array}\right\} \quad (579)$$

To find Δn, which depends upon the corrections of x' and y', we observe that x' and y', regarded as derivatives of x and y, are of the form

$$x' = \frac{dX}{dT} . \frac{1}{\sin \pi} \qquad\qquad y' = \frac{dY}{dT} . \frac{1}{\sin \pi}$$

But $\dfrac{dX}{dT}$ and $\dfrac{dY}{dT}$ depend upon the changes of the moon's right ascension and declination, which for the brief duration of an eclipse are correctly given in the Ephemeris. The errors of x' and y', therefore, depend upon those of π: so that if we write

$$x' = \frac{a}{\sin \pi} \qquad\qquad y' = \frac{b}{\sin \pi}$$

and regard a and b as correct, we find

$$\Delta x' = - x'\frac{\Delta\pi}{\tan \pi} \qquad\qquad \Delta y = - y'\frac{\Delta\pi}{\tan \pi}$$

From the equations $n \sin N = x'$, $n \cos N = y'$, we have

$$\Delta n \sin N + \Delta N \cos N = \Delta x' = - n \sin N . \frac{\Delta\pi}{\tan \pi}$$

$$\Delta n \cos N - \Delta N \sin N = \Delta y' = - n \cos N . \frac{\Delta\pi}{\tan \pi}$$

whence, by eliminating ΔN,*

$$\frac{\Delta n}{n} = -\frac{\Delta \pi}{\tan \pi} \qquad (580)$$

Since $\Delta(\alpha - a)$, $\Delta(\delta - d)$, $\Delta \pi$ will in practice be expressed in seconds of arc, we should substitute for them $\Delta(\alpha - a)\sin 1''$, $\Delta(\delta - d)\sin 1''$, $\Delta \pi \sin 1''$ in the above expressions; but if we at the same time put $\pi \sin 1''$ for $\sin \pi$ and $\tan \pi$, the factor $\sin 1''$ will disappear.

To develop ΔL, we may neglect the error of the small term $i\zeta$ and assume $\Delta L = \Delta l$. We have from (486) and (488), by neglecting the small term $k \sin \pi_0$ and putting $g = 1$, $z = \dfrac{1}{\sin \pi}$, the following approximate expression for l:

$$l = \frac{\sin H}{r' \sin \pi} \pm k$$

which gives

$$\Delta L = \Delta l = \frac{\Delta H}{r' \pi} \pm \Delta k - \frac{H}{r' \pi} \cdot \frac{\Delta \pi}{\pi} \qquad (581)$$

Substituting the values of ΔT_1, Δx, Δn, and Δl given by (579), (580), and (581), in (577), and putting

$$\nu = \frac{h}{n\pi}$$

the formula becomes, finally,

$$\Delta \omega = -\nu \left[\quad \sin N \cos d.\Delta(\alpha - a) + \cos N.\Delta(\delta - d) \right]$$

$$+ \nu \left[-\cos N \cos \delta.\Delta(\alpha - a) + \sin N.\Delta(\delta - d) \right] \tan \psi$$

$$+ \nu \left[\frac{\Delta H}{r'} \pm \pi \Delta k \right] \sec \psi$$

$$+ \nu \left[n (t + \omega - T_1) - x \tan \psi - \frac{H}{r' \pi} \sec \psi \right] \Delta \pi$$

$$+ \nu \left[\tfrac{1}{2} \beta\beta \left[n (t + \omega - T_1) - x \tan \psi - L \sec \psi \right] - \frac{\beta \cos d \cos (N + \psi)}{\cos \psi} \right] \pi \Delta e e$$

$$\qquad\qquad (582)$$

where the negative sign of $\pi \Delta k$ is to be used for interior contacts.

It is easily seen that $\pi \Delta k$ represents very nearly the correction

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* The angle N is independent of errors in π, since $\tan N = \dfrac{a}{b}$: so that we might have taken $\Delta N = 0$.

of the moon's apparent semidiameter, and $\dfrac{\Delta H}{r'}$ that of the sun's semidiameter; and that $\pi \Delta ee$ is the correction of the assumed reduction of the parallax for the latitude 90°.

334. *Discussion of the equations of condition for the correction of the longitude and of the elements of the computation.*—The longitude ω found by the equation (570), (Art. 329), requires the correction $\Delta \omega$ of (582). If, for brevity, we put

$$\left. \begin{array}{l} \gamma = \quad \sin N \cos \delta \, \Delta(\alpha - a) + \cos N \Delta(\delta - d) \\ \vartheta = - \cos N \cos \delta \, \Delta(\alpha - a) + \sin N \Delta(\delta - d) \end{array} \right\} \quad (583)$$

and

$$\omega' = \text{the true longitude,}$$

we have the equation of condition

$$\omega' = \omega + \Delta \omega = \omega - \nu \gamma + \nu \tan \psi \cdot \vartheta + \&c. \qquad (584)$$

If the eclipse has been observed at several places, we can form as many such equations as there are contacts observed. If the observations are complete at all the places, we can, for the most part, eliminate from these equations the unknown corrections of the elements, and determine the *relative* longitudes of the several places; and if the absolute longitude of one of the places is known, that of each place will also be determined.

I shall at first consider only the terms involving γ and ϑ. The quantity $\nu \gamma$ is a constant for all the places of observation, and combines with ω, so that it cannot be determined unless the longitude of at least one of the places is known. If then we put

$$\Omega = \omega' + \nu \gamma \qquad\qquad a = \nu \tan \psi$$

the equations of condition will assume the form

$$\Omega - a \vartheta - \omega = 0$$

Suppose, for the sake of completeness, that the four contacts of a total or annular eclipse have been observed at any one place, and that the values of the longitude found from the several contacts by Art. 329 are $\omega_1, \omega_2, \omega_3, \omega_4$. We then have the four equations

$$\begin{array}{ll} [1] & \Omega - a_1 \, \vartheta - \omega_1 = 0 \\ [2] & \Omega - a_2 \, \vartheta - \omega_2 = 0 \\ [3] & \Omega - a_3 \, \vartheta - \omega_3 = 0 \\ [4] & \Omega - a_4 \, \vartheta - \omega_4 = 0 \end{array}$$

where the numerals may be assumed to express the order in which the contacts are observed; [1] and [4] being exterior, and [2] and [3] interior. In a partial eclipse we should have but the 1st and 4th of these equations.

Since exterior contacts cannot (in most cases) be observed with as much precision as interior ones, let us assign different weights to the observations, and denote them by p_1, p_2, p_3, p_4, respectively. Combining the four equations according to the method of least squares, we form the two normal equations

$$[p \ ]\, \Omega - [pa \ ]\, \vartheta - [p\omega \ ] = 0$$
$$[pa]\, \Omega - [paa]\, \vartheta - [pa\omega] = 0$$

where the rectangular brackets are used as symbols of summation. From these, by eliminating Ω, and putting

$$[paa] - \frac{[pa]}{[p]}[pa] = P$$

$$[pa\omega] - \frac{[p\omega]}{[p]}[pa] = Q$$

we find

$$P\vartheta + Q = 0 \tag{585}$$

from which the value of ϑ would be determined with the weight P. But the computation of Q under this form is inconvenient. By developing the quantities P and Q, observing that $[paa] = p_1 a_1^2 + p_2 a_2^2 + p_3 a_3^2 + p_4 a_4^2$, &c., we shall find

$$P = \frac{p_1 p_2 (a_1 - a_2)^2 + p_1 p_3 (a_1 - a_3)^2 + p_1 p_4 (a_1 - a_4)^2}{p_1 + p_2 + p_3 + p_4}$$
$$+ \frac{p_2 p_3 (a_2 - a_3)^2 + p_2 p_4 (a_2 - a_4)^2 + p_3 p_4 (a_3 - a_4)^2}{p_1 + p_2 + p_3 + p_4}$$

$$Q = \frac{p_1 p_2 (a_1 - a_2)(\omega_1 - \omega_2) + p_1 p_3 (a_1 - a_3)(\omega_1 - \omega_3) + p_1 p_4 (a_1 - a_4)(\omega_1 - \omega_4)}{p_1 + p_2 + p_3 + p_4}$$
$$+ \frac{p_2 p_3 (a_2 - a_3)(\omega_2 - \omega_3) + p_2 p_4 (a_2 - a_4)(\omega_2 - \omega_4) + p_3 p_4 (a_3 - a_4)(\omega_3 - \omega_4)}{p_1 + p_2 + p_3 + p_4}$$

These forms show that if we subtract each of the equations [1], [2], [3] from each of those that follow it in the group, whereby we obtain the six equations

$$(a_1 - a_2)\, \vartheta + \omega_1 - \omega_2 = 0$$
$$(a_1 - a_3)\, \vartheta + \omega_1 - \omega_3 = 0$$
$$(a_1 - a_4)\, \vartheta + \omega_1 - \omega_4 = 0$$
$$(a_2 - a_3)\, \vartheta + \omega_2 - \omega_3 = 0$$
$$(a_2 - a_4)\, \vartheta + \omega_2 - \omega_4 = 0$$
$$(a_3 - a_4)\, \vartheta + \omega_3 - \omega_4 = 0$$

that is, it is simply the sum of all the individual equations in ϑ formed for the places severally.

The same reasoning is applicable to any of the terms which follow the term in ϑ in (584); so that if we suppose all the terms to be retained, this process gives an equation in ϑ for each place, in which besides the term $P\vartheta$ there will be terms in Δk, ΔH, &c., and from all the equations, by addition, a final normal equation (still called the equation in ϑ) as before. In the same manner, final normal equations in Δk, ΔH, &c. will be formed. Thus we shall obtain five normal equations involving the five unknown quantities ϑ, Δk, ΔH, $\Delta\pi$, Δee, which are then determined by solving the equations in the usual manner. But, unless the eclipse has been observed at places widely distant in longitude, it will not be possible to determine satisfactorily the value of $\Delta\pi$, much less that of Δee. It will be advisable to retain these terms in our equations, however, in order to show what effect an error in π or ee may produce upon the resulting longitudes.

When ϑ, &c. have been found, we find $\mathit{\Omega}$, $\mathit{\Omega}'$, &c. from the equations [1], [2] .... [5], [6] .... The final value of $\mathit{\Omega}$ will be the mean of its values [1 — 4] taken with regard to the weights; and so of $\mathit{\Omega}'$, &c. Hence we shall know the several *differences* of longitude

$$\omega' - \omega'' = \mathit{\Omega} - \mathit{\Omega}', \qquad \omega' - \omega''' = \mathit{\Omega} - \mathit{\Omega}'', \text{ \&c.}$$

If one of the longitudes, as for instance ω', is previously known, we have

$$\nu\gamma = \mathit{\Omega} - \omega'$$

and hence all the longitudes become known.

Finally, from the values of γ and ϑ the corrections of the Ephemeris in right ascension and declination are obtained by the formulæ

$$\left.\begin{array}{l} \cos\delta\Delta(a - a) = \sin N \cdot \gamma - \cos N \cdot \vartheta \\ \Delta(\delta - d) = \cos N \cdot \gamma + \sin N \cdot \vartheta \end{array}\right\} \quad (586)$$

335. When only two places of observation are considered, one of which is known, it will be sufficiently accurate to deduce γ and ϑ from the observations at the known place (disregarding the other corrections), and to employ their values in finding the longitude of the other place.

and y' for the time $t + \omega$. Compute the auxiliaries m, M, &c. by the formulæ*

$$m \sin M = x_0 - \xi \qquad\qquad n \sin N = x'$$
$$m \cos M = y_0 - \eta \qquad\qquad n \cos N = y'$$
$$\sin \psi = \frac{m \sin (M - N)}{L}$$

where ψ is (in general) to be so taken that $L \cos \psi$ shall be negative for a first and positive for a last contact (but in certain exceptional cases of rare occurrence see Art. 330).

Then

$$\tau = \frac{h L \cos \psi}{n} - \frac{h m \cos (M - N)}{n}$$

or, when $\sin \psi$ is not very small,

$$\tau = \frac{h m}{n} \cdot \frac{\sin (M - N - \psi)}{\sin \psi}$$

If the local mean time t was observed, take $h = 3600$ in these formulæ, and then the (uncorrected) longitude is found by the equation

$$\omega = T_0 - t + \tau$$

If the local sidereal time μ was observed, take $h = 3609.856$, in the preceding formulæ; then, μ_0 being the sidereal time at the first meridian corresponding to T_0, we have

$$\omega = \mu_0 - \mu + \tau$$

The longitudes thus found will be the true ones only when all the elements of the computation are correct.

IV. To form the equations of condition for the correction of these longitudes, when the eclipse has been observed at a sufficient number of places, compute the time T_1 of nearest approach, and the minimum distance x, by the formulæ

$$T_1 = T_0 - \frac{1}{n} (x_0 \sin N + y_0 \cos N)$$
$$x = - x_0 \cos N + y_0 \sin N$$

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* The values of N and $\log n$ being nearly constant, it will be expedient, where many observations are to be reduced, to compute them for the several integral hours at the first meridian, and to deduce their values for any given time by simple interpolation.

For these places we have given—

	Lat. ϕ	Long. ω
Washington,	$+ 38° 53' 39''.25$	$+ 5^h 8^m 11^s.2$
Königsberg,	$+ 54\ 42\ 50\ .4$	$- 1\ 22\ 0\ .4$

The longitudes are reckoned from Greenwich. That of Königsberg will be assumed as correct, while that of Washington will be regarded as an approximate value which it is proposed to correct by these observations.

I. The mean Greenwich time of conjunction of the sun and moon in right ascension being, July 28, $2^h 21^m 2^s.6$, the general eclipse tables will be constructed for the Greenwich hours 0^h, 1^h, 2^h, 3^h, 4^h, and 5^h of July 28. For these times we find the following quantities from the *Nautical Almanac:*

For the Moon.*

Greenwich mean time.	a	δ	π
July 28, 0^h	$125° 40'\ 6''.75$	$+ 20°\ 3' 30''.00$	$60' 27''.30$
1	$126\ 19\ 9\ .41$	$19\ 58\ 9\ .36$	$28\ .41$
2	$126\ 58\ 10\ .80$	$19\ 52\ 39\ .99$	$29\ .49$
3	$127\ 37\ 10\ .82$	$19\ 47\ 1\ .92$	$30\ .54$
4	$128\ 16\ 9\ .37$	$19\ 41\ 15\ .21$	$31\ .56$
5	$128\ 55\ 6\ .36$	$19\ 35\ 19\ .89$	$32\ .56$

For the Sun.

Greenwich mean time.	a'	δ'	$\log r'$
July 28, 0^h	$127°\ 6'\ 5''.25$	$+ 19° 5' 24''.70$	0.006578
1	$8\ 32\ .63$	$4\ 50\ .23$	76
2	$10\ 59\ .99$	$4\ 15\ .74$	74
3	$13\ 27\ .34$	$3\ 41\ .21$	72
4	$15\ 54\ .67$	$3\ 6\ .64$	70
5	$18\ 21\ .99$	$2\ 32\ .05$	67

* The moon's a and δ in the *Naut. Alm.* are directly computed only for every noon and midnight and interpolated for each hour. I have not used these interpolated values, but have interpolated anew to fifth differences. The moon's parallax has been diminished by $0''.3$ according to Mr. ADAMS's Table in the Appendix to the *Naut. Alm.* for 1856.

With these values we form the following tables, as in Art. 297:

	a	d	Exterior Contacts.		Interior Contacts.	
			l	log i	l	log i
0ʰ	127° 6′ 17″.22	19° 5′ 16″.56	0.584046	7.663244	— 0.011771	7.661131
1	8 39 .51	4 42 .76	4023	45	11795	32
2	11 1 .78	4 8 .96	3978	47	11844	34
3	13 24 .03	3 35 .14	3899	49	11917	36
4	15 46 .27	3 1 .30	3801	51	12015	38
5	18 8 .50	2 27 .46	3679	53	12187	40

	x	Δ_1	Δ_2	Δ_3	y	Δ_1	Δ_2	Δ_3
0ʰ	— 1.338900				+ 0.968589			
1	— 0.769366	+ 0.569584	+ 57	—58	.885569	— 0.083020	—364	
2	— 0.199775	.569591	— 1	—54	.802185	.083384	—352	+12
3	+ 0.369815	.569590	— 55	—64	.718449	.083736	—343	+ 9
4	+ 0.939350	.569535	—119		.634370	.084079	—341	+ 2
5	+ 1.508766	.569416			.549950	.084420		

Hence the mean changes x' and y', for the epoch $T_0 = 2^h$ (according to the method of Art. 296), and the corresponding values of N and log n, are as follows:

	x'	y'	N	log n
0ʰ	+ 0.569563	— 0.083202	98° 18′ 39″.7	9.760126
1	591	3384	19 42 .7	168
$T_0 = 2$	600	3562	20 45 .3	194
3	590	3736	21 47 .5	205
4	563	3908	22 50 .0	203
5	514	4078	23 52 .7	186

II. The full computation for Königsberg, where both exterior and interior contacts were observed, will serve to illustrate the use of the preceding formulæ in every practical case.

For $\varphi = 54° 42′ 50″.4$ we find

$$\log \rho \sin \varphi' = 9.909898 \qquad \log \rho \cos \varphi' = 9.762639$$

The sidereal time at Greenwich mean noon, July 28, was $8^h 22^m 13^s.27$, with which μ is found as given below. The computation of ξ, η, and L will be as follows:

	1st Ext. Cont.	1st Int. Cont.	2d Int. Cont.	2d Ext. Cont.
t	$8^h\ 38^m\ 10^s.8$	$4^h\ 38^m\ 57^s.6$	$4^h\ 41^m\ 54^s.2$	$5^h\ 38^m\ 32^s.9$
$t + \omega$	2 16 10.4	3 16 57.2	3 19 53.8	4 16 32.5
μ	12 0 46.44	13 1 43.22	13 4 40.31	14 1 28.31
μ (in arc)	180° 11′ 36″.6	195° 25′ 48″.3	196° 10′ 4″.7	210° 22′ 4″.7
For $t + \omega$, a	127 11 40 .1	127 14 4 .2	127 14 11 .2	127 16 25 .6
" d	19 3 59 .8	19 3 25 .6	19 3 28 .9	19 2 52 .0
$\mu - a$	52 59 56 .5	68 11 44 .1	68 55 58 .5	83 5 39 .1
log sin $(\mu - a)$	9.902343	9.967762	9.969952	9.996838
log cos $(\mu - a)$	9.779473	9.569889	9.555679	9.080040
log ξ	9.664982	9.780401	9.782591	9.759477
ξ	$+ 0.462362$	$+ 0.537528$	$+ 0.540244$	$+ 0.574748$
log A sin B	9.909898	9.909898	9.909898	9.909898
log A cos B	9.542112	9.332528	9.318318	8.842679
B	66° 47′ 32″.2	75° 10′ 40″.4	75° 38′ 5″.9	85° 6′ 14″.3
$B - d$	47 43 32 .4	56 7 14 .8	56 34 42 .0	66 3 22 .3
log A	9.946544	9.924595	9.923693	9.911486
log sin $(B - d)$	9.869192	9.919191	9.921499	9.960919
log cos $(B - d)$	9.827809	9.746201	9.740991	9.608355
log η	9.815736	9.843786	9.845192	9.872405
η	$+ 0.654239$	$+ 0.697888$	$+ 0.700152$	$+ 0.745427$
log ζ	9.774353	9.670796	9.664684	9.519841
For $t + \omega$, log i	7.663248	7.661137	7.661187	7.663252
" " l	$+ 0.533956$	$- 0.011940$	$- 0.011944$	$+ 0.533772$
$i\zeta$	$+ 0.002739$	$+ 0.002148$	$+ 0.002117$	$+ 0.001524$
L	$+ 0.531217$	$- 0.014088$	$- 0.014061$	$+ 0.532248$

III. The epoch of the table of x' and y' being $T_0 = 2^h$, we have for this time

$$x_0 = - 0.199775 \qquad y_0 = + 0.802185$$

with which we proceed to find the values of ω.

m sin $M = x_0 - \xi$	$- 0.662137$	$- 0.737303$	$- 0.740019$	$- 0.774523$
m cos $M = y_0 - \eta$	$+ 0.147946$	$+ 0.104297$	$+ 0.102033$	$+ 0.056758$
log m sin M	$n9.820948$	$n9.867646$	$n9.869242$	$n9.889035$
log m cos M	9.170107	9.018272	9.008741	8.754027
M	282° 35′ 42″.8	278° 3′ 5″.4	277° 51′ 1″.5	274° 11′ 28″.3
log m	9.831527	9.871949	9.873331	9.890198
For $t + \omega$, N	98° 21′ 2″.1	98° 22′ 5″.1	98° 22′ 8″.2	98° 23′ 7″.3
" " log n	9.760198	9.760206	9.760206	9.760200
$M - N$	184° 14′ 40″.7	179° 41′ 0″.3	179° 28′ 53″.3	175° 48′ 21″.0
log sin $(M - N)$	$n8.869321$	7.742363	7.956643	8.864135

$\log L$	9.725272	n8.148849	n8.148016	9.726114
$\log \sin \downarrow$	n8.975576	n9.465463	n9.681958	9.028219
$\downarrow$	185° 25′ 27″.7	343° 1′ 8″.6	208° 44′ 14″.0	6° 7′ 33″.2
$M - N - \downarrow$	358 49 13 .0	196 39 51 .7	330 44 39 .3	169 40 47 .8
$\log \sin (M - N - \downarrow)$	n8.313626	n9.457526	n9.689051	9.253208
$h = 8600, \quad \log h$	8.556303			
$\log \tau$	2.965682	8.660109	8.676521	3.911290
τ	$+ 0^h 15^m 24^s.0$	$+ 1^h 16^m 12^s.0$	$+ 1^h 19^m 8^s.1$	$+ 2^h 15^m 52^s.5$
$T_0 - t$	$- 1\ 38\ 10\ .8$	$- 2\ 38\ 57\ .6$	$- 2\ 41\ 54\ .2$	$- 3\ 38\ 32\ .9$
ω	$- 1\ 22\ 46\ .8$	$- 1\ 22\ 45\ .6$	$- 1\ 22\ 46\ .1$	$- 1\ 22\ 40\ .4$

IV. *Equations of condition.*—To find T_1 and x, we have for $T_0 = 2^h$,

$$\log x_0 = n9.3006 \qquad\qquad N = 98°\ 20'.7$$
$$\log y_0 = \quad 9.9043 \qquad\qquad \log n = 9.7602$$

whence

$$-\frac{x_0 \sin N}{n} = + 0.3434 \qquad - x_0 \cos N = -0.0290 \quad \log H = 2.9822$$

$$-\frac{y_0 \cos N}{n} = + 0.2023 \qquad + y_0 \sin N = + 0.7938 \quad \log r' = 0.0066$$

$$T_1 = \quad 2^h.5457 \qquad\qquad x = + 0.7648 \quad \log \pi = 3.5599$$

$$\pi = 3630'' \qquad\qquad \log x = \quad 9.8835 \quad \log \frac{h}{r'\pi} = 9.4157$$

$$\log \beta = \log \frac{\rho \sin \varphi'}{1 - ee} = 9.9128 \qquad\qquad \log \nu = \log \frac{h}{n\pi} = 0.2362$$

With these constants prepared, we readily form the coefficients of the equations of condition as follows:

	1st Ext. Cont.	1st Int. Cont.	2d Int. Cont.	2d Ext. Cont.
$\log \tan \downarrow$	8.9775	n9.4848	9.7390	9.0307
$\log \sec \downarrow$	n0.0019	0.0194	n0.0571	0.0025
$\nu \tan \downarrow$	$+ 0.163$	$- 0.526$	$+ 0.944$	$+ 0.185$
$\nu \sec \downarrow$	$- 1.730$	$+ 1.801$	$- 1.964$	$+ 1.733$
$t + \omega - T_1$	$- 0^h.2762$	$+ 0^h.7355$	$+ 0^h.7860$	$+ 1^h.7300$
$\log (t + \omega - T_1)$	n9.4412	9.8666	9.8954	0.2380
$\nu n (t + \omega - T_1)$	$- 0.2739$	$+ 0.7295$	$+ 0.7795$	$+ 1.7155$
$- x\nu \tan \downarrow$	$- 0.1251$	$+ 0.4023$	$- 8.7223$	$- 0.1414$
$- \dfrac{H}{r'\pi} \nu \sec \downarrow$	$+ 0.4506$	$- 0.4691$	$+ 0.5117$	$- 0.4512$
E	$+ 0.0516$	$+ 0.6627$	$+ 0.5689$	$+ 1.1229$

	1st Ext. Cont.	1st Int. Cont.	2d Int. Cont.	2d Ext. Cont.
$\nu n\,(t + \omega - T_1)$	− 0.2789	+ 0.7295	+ 0.7795	+ 1.7155
− $x\nu\tan\psi$	− 0.1251	+ 0.4023	− 0.7223	− 0.1414
− $L\nu\sec\psi$	+ 0.9192	+ 0.0254	− 0.0276	− 0.9222
	+ 0.5202	+ 1.1572	+ 0.0296	+ 0.6519
log	9.7162	0.0634	8.4713	9.8142
log $\frac{1}{2}\beta\beta$	9.5246			
log 1st part of F	9.2408	9.5880	7.9959	9.3388
$N + \psi$	283° 46'.2	81° 21'.8	307° 4'.9	104° 28'.3
log cos $(N + \psi)$	9.3766	9.1766	9.7803	n9.8978
log $(-\nu\beta\cos d\sec\psi)$	0.1264	n0.1439	0.1816	n0.1270
log 2d part of F	9.5030	n9.3205	9.9619	9.5248
1st part of F	+ 0.1741	+ 0.3873	+ 0.0099	+ 0.2182
2d " " F	+ 0.3184	− 0.2092	+ 0.9160	+ 0.3849
F	+ 0.4925	+ 0.1781	+ 0.9259	+ 0.5531

Putting $\omega' + \nu\gamma = \mathit{\Omega}$, we have, therefore, for the four Königsberg observations, the equations

$$
\text{(A)}\left|\begin{array}{c} \frac{p}{} \\ 1 \\ 2 \\ 2 \\ 1 \end{array}\right|
\begin{array}{l}
\mathit{\Omega} = -1^h\,22^m\,46^s.8 + 0.163\,\vartheta - 1.780\,\pi\Delta k - 1.780\,\dfrac{\Delta H}{r'} + 0.052\,\Delta\pi + 0.498\,\pi\Delta ee \\
\mathit{\Omega} = -1\ \ 22\ \ 45\ .6 - 0.526\ \ \ -1.801\ \ \ \ \ +1.801\ \ \ \ \ +0.663\ \ \ \ \ +0.178 \\
\mathit{\Omega} = -1\ \ 22\ \ 46\ .1 + 0.944\ \ +1.964\ \ \ \ \ -1.964\ \ \ \ \ +0.569\ \ \ \ \ +0.926 \\
\mathit{\Omega} = -1\ \ 22\ \ 40\ .4 + 0.185\ \ +1.733\ \ \ \ \ +1.733\ \ \ \ \ +1.123\ \ \ \ \ +0.553
\end{array}
$$

where we have annexed a column for the weight p, giving interior contacts double weight.

A similar computation for the two observations at Washington gives the following equations, in which $\mathit{\Omega}' = \omega'' + \nu\gamma$, ω'' denoting the true longitude of Washington:

$$
\text{(B)}\left|\begin{array}{c} \frac{p}{} \\ 1 \\ 1 \end{array}\right|
\begin{array}{l}
\mathit{\Omega}' = 5^h\,7^m\,29^s.9 + 1.660\,\vartheta - 2.392\pi\Delta k - 2.892\,\dfrac{\Delta H}{r'} - 2.681\,\Delta\pi + 0.722\,\pi\Delta ee \\
\mathit{\Omega}' = 5\ \ 7\ \ 21\ .9 - 2.406\ \ \ +2.959\ \ \ \ \ +2.959\ \ \ \ \ +0.509\ \ \ \ \ -1.323
\end{array}
$$

More observations would be necessary in order to determine all the corrections; but I shall retain all the terms in order to illustrate the general method. Subtracting each of the Königsberg equations from each of those which follow it, we obtain the six equations,

$$\text{(A')} \begin{array}{c|c} & \left|\dfrac{p}{-}\right| \\ \hline \frac{1}{3} & 0 = + 1^{s}.2 - 0.689\,\vartheta - 0.071\,\pi\Delta k + 3.531\,\dfrac{\Delta H}{r'} + 0.611\,\Delta\pi - 0.315\,\pi\Delta ee \\ \frac{1}{4} & 0 = + 0.7 + 0.781 + 8.694 - 0.234 + 0.517 + 0.433 \\ \frac{1}{6} & 0 = + 6.4 + 0.022 + 3.463 + 8.463 + 1.071 + 0.060 \\ \frac{2}{3} & 0 = - 0.5 + 1.470 + 3.765 - 3.765 - 0.094 + 0.748 \\ \frac{1}{6} & 0 = + 5.2 + 0.711 + 3.534 - 0.068 + 0.460 + 0.375 \\ \frac{1}{3} & 0 = + 5.7 - 0.759 - 0.231 + 3.697 + 0.554 - 0.373 \end{array}$$

where the weight in each case is the quotient obtained by dividing the product of the two weights of the equations whose difference is taken, by the sum of the weights of the four original equations (Art. 334).

The same method, applied in the case of the two Washington equations, gives the single equation

$$\text{(B')} \left|\dfrac{p}{\frac{1}{4}}\right| \quad 0 = - 8^{s}.0 - 4.066\,\vartheta + 5.851\,\pi\Delta k + 5.351\,\dfrac{\Delta H}{r'} + 8.190\,\Delta\pi - 2.055\,\pi\Delta ee$$

From the equations (A') and (B') are formed the following final equations, having regard to their weights, in the usual manner:

$$\begin{aligned}
0 &= + 15.495 + 10.426\,\vartheta - 5.300\,\pi\Delta k - 16.377\,\tfrac{\Delta H}{r'} - 6.609\,\Delta\pi + 5.281\,\pi\Delta ee \\
0 &= - 12.445 - 5.300 + 34.506 + 6.185 + 10.040 - 2.575 \\
0 &= - 8.191 - 16.377 + 6.185 + 34.505 + 10.740 - 8.214 \\
0 &= - 9.371 - 6.609 + 10.040 + 10.740 + 5.672 - 3.316 \\
0 &= + 7.951 + 5.281 - 2.575 - 8.214 - 3.316 + 2.675
\end{aligned}$$

As we cannot expect a satisfactory determination of $\Delta\pi$ and $\pi\Delta ee$ from these observations, we disregard the last two equations, and then, solving the first three, we obtain ϑ, $\pi\Delta k$, and $\dfrac{\Delta H}{r'}$ in terms of $\Delta\pi$ and $\pi\Delta ee$, as follows:

$$\begin{aligned}
\vartheta &= - 4''.36 + 0.375\,\Delta\pi - 0.525\,\pi\Delta ee \\
\pi\Delta k &= + 0.02 - 0.216\,\Delta\pi - 0.004\,\pi\Delta ee \\
\frac{\Delta H}{r'} &= - 1.83 - 0.095\,\Delta\pi - 0.010\,\pi\Delta ee
\end{aligned}$$

These values substituted in the equations (A) give

$$\begin{aligned}
\Omega &= - 1^{h}\,22^{m}\,44^{s}.38 + 0.651\,\Delta\pi + 0.432\,\pi\Delta ee \\
\Omega &= - 1\,22\,46.64 + 0.684 + 0.443 \\
\Omega &= - 1\,22\,46.58 + 0.685 + 0.442 \\
\Omega &= - 1\,22\,44.34 + 0.653 + 0.432
\end{aligned}$$

the mean of which, giving the second and third double weight, is

(A'') $\qquad \mathcal{Q} = -1^h 22^m 45^s.86 + 0.674 \, \Delta\pi + 0.439 \, \pi\Delta ee$

The equations (B) become

$$\mathcal{Q}' = 5^h 7^m 26^s.99 - 1.314 \, \Delta\pi - 0.116 \, \pi\Delta ee$$
$$\mathcal{Q}' = 5 \ \ 7 \ \ 27.03 - 1.314 \qquad - 0.101$$

the mean of which is

(B'') $\qquad \mathcal{Q}' = 5^h 7^m 27^s.01 - 1.314 \, \Delta\pi - 0.109 \, \pi\Delta ee$

Now, if we assume the longitude of Königsberg to be well determined, we have

$$\mathcal{Q} = \omega' + \nu\gamma = -1^h 22^m 0^s.4 + \nu\gamma$$

which, with the equation (A''), gives

$$\nu\gamma = -45^s.46 + 0.674 \, \Delta\pi + 0.439 \, \pi\Delta ee$$

Hence, by (B''), the true longitude of Washington is

$$\omega'' = \mathcal{Q}' - \nu\gamma = 5^h 8^m 12^s.47 - 1.988 \, \Delta\pi - 0.548 \, \pi\Delta ee$$

If the longitude of Washington were also previously well established, this last equation would give us a condition for determining the correction of the moon's parallax. Thus, if we adopt $\omega'' = 5^h 8^m 12^s.34$, which results from the U. S. Coast Survey Chronometric Expeditions of 1849, '50, '51, and '55, this equation gives

$$0 = +0.13 - 1.988 \, \Delta\pi - 0.548 \, \pi\Delta ee$$

whence

$$\Delta\pi = +0''.07 - 0.276 \, \pi\Delta ee$$

The probable value of Δee, according to BESSEL, is within ± 0.0001, so that the last term cannot here exceed $0''.10$. If, therefore, the above observations are reliable and the supposed longitudes exact, the probable correction of the parallax indicated scarcely exceeds $0''.1$, a quantity too small to be established by so small a number of observations. Nevertheless, the example proves both that the adopted parallax is very nearly perfect, and that a large number of observations at various well determined places in the two hemispheres may furnish a good determination of the correction which it yet requires.

Finally, the corrections of the Ephemeris in right ascension and declination, according to the above determination of γ and ϑ, are found by (586) to be (putting a' for a and δ' for d)

$$\Delta(a - a') = -28''.42 + 0.469\, \Delta\pi + 0.187\,\pi\Delta ee$$
$$\Delta(\delta - \delta') = -0.48 + 0.314\, \Delta\pi - 0.556\,\pi\Delta ee$$

This large correction in right ascension agrees with the results of the best meridian observations on and near the date of this eclipse. Since that time the Ephemerides have been greatly improved by the use of HANSEN's new Tables of the Moon.

LUNAR ECLIPSES.

338. *To find whether near a given opposition of the moon and sun a lunar eclipse will occur.*—The solution of this problem is similar to that of Art. 287, except that for the sun's semidiameter there must be substituted the apparent semidiameter of the earth's shadow at the distance of the moon; and also that the apparent distance of the centres of the moon and the shadow will not be affected by parallax, since when the earth's shadow falls upon the moon an eclipse occurs for all observers who have the moon above their horizon. If S, Fig. 45, is the sun's centre, E that of the earth, LM the semidiameter of the earth's shadow at the moon, we have

Fig. 45.
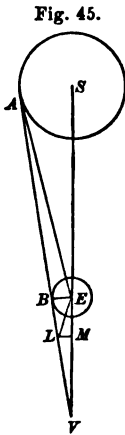
Apparent semidiameter of the total
shadow $= LEM$
$= BLE - EVL$
$= BLE - (AES - EAV)$
$= \pi - s' + \pi'$

where we employ the same notation as in Art. 287.

But observation has shown that the earth's atmosphere increases the apparent breadth of the shadow by about its one-fiftieth part:* so that we take

---

* This fractional increase of the breadth of the shadow was given by LAMBERT as $\frac{1}{40}$, and by MAYER as $\frac{1}{60}$. BEER and MÄDLER found $\frac{1}{50}$ from a number of observations of eclipses of lunar spots in the very favorable eclipse of December 26, 1833. See "*Der Mond nach seinen kosmischen und individuellen Verhältnissen, oder allgemeine vergleichende Selenographie, von* WILHELM BEER *und* DR. JOHANN HEINRICH MÄDLER," § 98.

$$\text{App. semid. of shadow} = \frac{51}{50}(\pi - s' + \pi') \qquad (587)$$

In order that a lunar eclipse may happen, we must have, therefore, instead of (477),

$$\beta \cos I' < \frac{51}{50}(\pi - s' + \pi') + s \qquad (588)$$

or, taking a mean value of I', as in Art. 287,

$$\beta < \left[\frac{51}{50}(\pi - s' + \pi') + s\right] \times 1.00472$$

Employing mean values in the small fractional part, we have

$$\left[\frac{51}{50}(\pi - s' + \pi') + s\right] \times .00472 = 16''$$

and the condition becomes

$$\beta < \frac{51}{50}(\pi - s' + \pi') + s + 16'' \qquad (589)$$

If in this we substitute the greatest values of π, π', and s, and the least value of s', the limit

$$\beta < 63' \, 53''$$

is the greatest limit of the moon's latitude at the time of opposition for which an eclipse can occur.

If we substitute the least values of π, π', and s, and the greatest value of s', the limit

$$\beta < 52' \, 4''$$

is the least limit for which an eclipse can fail to occur.

Hence, a lunar eclipse is *certain* if at full moon $\beta < 52' \, 4''$, *impossible* if $\beta > 63' \, 53''$, and doubtful between these limits. The doubtful cases can be examined by (589), or still more exactly by (588), employing the actual values of π, π', s, s', at the time, and computing I' by (475).

These limits are for the total shadow. For the penumbra we have

$$\text{App. semid. of penumbra} = \frac{51}{50}(\pi + s' + \pi') \qquad (590)$$

so that the condition (588) may be employed to determine whether any portion of the penumbra will pass over the moon, by substituting $+ s'$ for $- s'$. It will be worth while to make this examination only when it has been found that the total shadow does not fall upon the moon.

339. *To find the time when a given phase of a lunar eclipse will occur.*—The solution of this problem may be derived from the general formulæ given for solar eclipses, by interchanging the moon and earth and regarding the lunar eclipse as an eclipse of the sun seen from the moon; but the following direct investigation is even more simple.
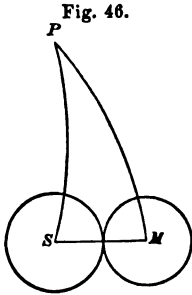
Fig. 46.

Let S, Fig. 46, be the point of the celestial sphere which is opposite the sun, or the apparent geocentric position of the centre of the earth's shadow; M, the geocentric place of the centre of the moon; P, the north pole. If we put

$\alpha =$ the right ascension of the moon,
$\alpha' =$ the right ascension of the point S,
$=$ R. A. of the sun $+ 180°$,
$\delta =$ the declination of the moon,
$\delta' =$ the declination of the sun,
$Q =$ the angle PSM,
$L = SM$,

we have

$- \delta' =$ the declination of S,

and the triangle PSM gives

$$\left. \begin{array}{l} \sin L \sin Q = \cos \delta \sin (\alpha - \alpha') \\ \sin L \cos Q = \cos \delta' \sin \delta + \sin \delta' \cos \delta \cos (\alpha - \alpha') \end{array} \right\} \quad (591)$$

The eclipse begins or ends when the arc SM is exactly equal to the sum of the apparent semidiameters of the moon and the shadow. The figure of the shadow will differ a little from a circle, as the earth is a spheroid; but it will be sufficiently accurate to regard the earth as a sphere with a mean radius, or that for the latitude 45°. This is equivalent to substituting for π in (587) and (590) the parallax reduced to the latitude 45°, which may be found by the formula

$$\pi_1 = [9.99929]\,\pi \qquad (592)$$

where the factor in brackets is given by its logarithm.

Hence the first and last contacts of the moon with the penumbra occur when we have

$$L = \frac{51}{50}(\pi_1 + s' + \pi') + s \qquad (593)$$

For the first and last contacts with the total shadow,

$$L = \frac{51}{50}(\pi_1 - s' + \pi') + s \qquad (594)$$

For the first and second internal contacts with the penumbra,

$$L = \frac{51}{50}(\pi_1 + s' + \pi') - s \qquad (595)$$

For the first and second internal contacts with the total shadow, or the beginning and end of total eclipse,

$$L = \frac{51}{50}(\pi_1 - s' + \pi') - s \qquad (596)$$

The solution of our problem consists in finding the time at which the equations (591) are satisfied when the proper value of L is substituted in them. A very precise computation would, however, be superfluous, as the contacts cannot be observed with accuracy, on account of the indefinite character of the outline both of the penumbra and of the total shadow. It will be sufficient to write for (591) the following approximate formulæ, easily deduced from them:

$$\left.\begin{array}{l} L\sin Q = (a - a')\cos \delta \\[2mm] L\cos Q = \delta + \delta' - \dfrac{\sin 2\delta \sin^2 \frac{1}{2}(a - a')}{\sin 1''} \end{array}\right\} \quad (597)$$

Let us put

$$\left.\begin{array}{l} \epsilon = \dfrac{\sin 2\delta \sin^2 \frac{1}{2}(a - a')}{\sin 1''} \\[2mm] x = (a - a')\cos \delta \\[1mm] y = \delta + \delta' - \epsilon \\[1mm] x', y' = \text{the hourly increase of } x \text{ and } y\,, \end{array}\right\} \quad (598)$$

then, if the values of x and y are computed for several successive

hours near the time of full moon, we shall also have x' and y' from their differences; and if x_0 and y_0 denote the values of x and y for an assumed epoch T_0, near the time of opposition, we shall have for the required time of contact $T = T_0 + \tau$ the equations

$$L \sin Q = x_0 + x'\tau$$
$$L \cos Q = y_0 + y'\tau$$

from which τ is obtained by the process already frequently employed in the preceding problems. Thus, computing the auxiliaries m, M, n, N, by the equations

$$m \sin M = x_0 \qquad\qquad n \sin N = x' \qquad \left.\right\} \quad (599)$$
$$m \cos M = y_0 \qquad\qquad n \cos N = y'$$

we shall have

$$\sin \psi = \frac{m \sin (M - N)}{L} \qquad\qquad \left.\right\}$$

$$\tau = \frac{L \cos \psi}{n} - \frac{m \cos (M - N)}{n} \qquad\qquad (600)$$

$$T = T_0 + \tau$$

in which we take $\cos \psi$ with the negative sign for the first contact and with the positive sign for the last contact.

The angle $Q = N + \psi$ is very nearly the supplement of the angle PMS, Fig. 46; from which we infer that *the angle of position of the point of contact reckoned on the moon's limb from the north point of the limb towards the east* $= 180° + N + \psi$.

The time of greatest obscuration is found, as in Art. 324, to be

$$T_1 = T_0 - \frac{m \cos (M - N)}{n} \qquad\qquad (601)$$

which is also the middle of the eclipse.

The least distance of the centres of the shadow and of the moon being denoted by $\varDelta$, we have, as in Art. 324,

$$\varDelta = \pm\, m \sin (M - N) \qquad\qquad (602)$$

the sign being taken so that $\varDelta$ shall be positive. If then we put

     $D =$ the magnitude of the eclipse, the moon's diameter being unity,

we evidently have

$$D = \frac{L - \varDelta}{2s} \qquad (603)$$

in which the value of L for total shadow from (594) is to be employed.

The small correction ϵ in (598) may usually be omitted, but its value may be taken at once from the following table:

Value of ϵ.

δ	$a - a'$						
	0″	1000″	2000″	3000″	4000″	5000″	6000″
0°	0″	0″	0″	0″	0″	0″	0″
5	0	0	1	2	3	5	8
10	0	0	2	4	7	10	15
15	0	1	2	6	10	15	22
20	0	1	3	7	13	19	28
25	0	1	4	8	15	23	33
30	0	1	4	9	17	26	38

The quantity ϵ has the same sign as δ, and is to be subtracted algebraically from $\delta + \delta'$.

EXAMPLE.—Compute the lunar eclipse of April 19, 1856. The Greenwich mean time of full moon is April 19, $21^h 5^m.5$. We therefore compute the co-ordinates x and y for the Greenwich times April 19, 18^h, 21^h, 24^h.

	18^h	21^h	24^h
☽ R. A. $= a$	$13^h 46^m 36^s.62$	$13^h 52^m 9^s.81$	$13^h 57^m 45^s.12$
☉ R.A. $+ 180° = a'$	13 52 52 .98	13 53 20 .93	13 53 48 .88
$a - a'$	$-$ 6 16 .36	$-$ 1 11 .12	$+$ 3 56 .24
$a - a'$ (in arc)	$-$ 5645″	$-$ 1067″	$+$ 3544″
☽ Decl. $= \delta$	$-11° 27' 0''.2$	$-12° 6' 43''.7$	$-12° 46' 5''.5$
☉ " $= \delta'$	$+11$ 35 49 .4	$+11$ 38' 22 .8	$+11$ 40 56 .6
$- \epsilon$	$+$ 13 .	0 .	$+$ 6.
y	$+$ 542″	$-$ 1701″	$-$ 3903″
$\log (a - a')$	$n3.75166$	$n3.02816$	3.54949
$\log \cos \delta$	9.99127	9.99022	9.98913
$\log x$	$n3.74293$	$n3.01838$	3.53862

Hence we have the following table:

	x	Diff. $= 3x'$			y	Diff. $= 3y'$	
18^h	$- 5533''$	$+4490$	$x' = + 1498$		$+ \;\; 542''$	$- 2243$	$y' = - 741$
21	$- 1043$	$+4499$			$- 1701$	$- 2202$	
24	$+ 3456$				$- 3903$		

To find L, we have, by (593) and (594),

$$\pi = 54' \, 32'' \qquad \pi_1 = 3267''$$
$$s' = \;\; 957$$
$$\pi' = \quad\; 9$$
$$\pi_1 - s' + \pi' = \overline{2319} \qquad\qquad\qquad \pi_1 + s' + \pi' = 4233''$$
$$\tfrac{1}{50}(\pi_1 - s' + \pi') = \quad 46 \qquad\qquad\qquad \tfrac{1}{50}(\pi_1 + s' + \pi') = \quad 85$$
$$s = \;\; 891 \qquad\qquad\qquad\qquad\qquad\qquad s = \;\; 891$$
$$L \text{ for shadow} = \overline{3256} \qquad\qquad\qquad L \text{ for penumbra} = \overline{5209}$$

Assuming the time $T_0 = 21^h$, we proceed by (599) and (600):

$x_0 = m \sin M$	$- 1043$		$x' = n \sin N$	$+ 1498$
$y_0 = m \cos M$	$- 1701$		$y' = n \cos N$	$- \;\; 741$
M	$210° \, 31'.0$		N	$116° \, 19'.2$
$\log m$	3.3000		$\log n$	3.2230

$$- \frac{m}{n} \cos (M - N) = + 0^h.108$$
$$T_0 = \quad 21$$
$$T_1 = \text{Time of middle of eclipse} = \quad \overline{21 \,.108}$$

	Shadow.	Penumbra.
$\log \sin \downarrow$	9.7855	9.5815
$\dfrac{L \cos \downarrow}{n}$	$\mp \; 1^h.542$	$\mp \; 2^h.881$
T_1	$21 \,.108$	$21 \,.108$
Beginning	$19 \,.566$	$18 \,.227$
End	$22 \,.650$	$23 \,.989$

For the magnitude of the eclipse, we have, by (602) and (603):

341. *To find the longitude from an observed occultation of a star by the moon.*—According to the method of Art. 329, we proceed as follows:

I. Find, approximately, the time of conjunction of the moon and star in right ascension, reckoned at the first meridian. Take from the Ephemeris, for four consecutive integral hours, two preceding and two following the time of conjunction, the moon's right ascension (α), declination (δ), and horizontal parallax (π). Take also from the most reliable source the star's right ascension (α') and declination (δ').

For each of these hours compute the co-ordinates x and y by the formulæ

$$x = \frac{\cos \delta \sin (\alpha - \alpha')}{\sin \pi}$$

$$y = \frac{\sin (\delta - \delta') \cos^2 \tfrac{1}{2} (\alpha - \alpha') + \sin (\delta + \delta') \sin^2 \tfrac{1}{2} (\alpha - \alpha')}{\sin \pi}$$

and, arranging their values in a table, deduce their hourly variations x' and y' for the same instants for which x and y have been computed.

II. Let μ be the local sidereal time of an observed immersion or emersion of the star at a place whose latitude is φ, and west longitude ω; t the corresponding local mean time. The co-ordinates of the place are to be computed by the formulæ

$$A \sin B = \rho \sin \varphi' \qquad\qquad \xi = \rho \cos \varphi' \sin (\mu - \alpha')$$
$$A \cos B = \rho \cos \varphi' \cos (\mu - \alpha') \qquad\qquad \eta = A \sin (B - \delta')$$
$$\zeta = A \cos (B - \delta')$$

When $\log \zeta$ is small, add to $\log \xi$ and η the correction for refraction from the table on p. 517.

III. Assume any convenient time T_0 reckoned at the first meridian, so near to $t + \omega$ that x and y may be considered to vary proportionally with the time in the interval $t + \omega - T_0$. For the assumed time, take the values of x and y (denoting them by x_0 and y_0), and also those of x' and y', and compute the auxiliaries m, M, &c. by the formulæ

$$m \sin M = x_0 - \xi \qquad n \sin N = x'$$
$$m \cos M = y_0 - \eta \qquad n \cos N = y'$$
$$\sin \psi = \frac{m \sin (M - N)}{k} \qquad \log .. = 9.435000*$$

where ψ is (in general) to be so taken that $\cos \psi$ shall be negative for immersion and positive for emersion (but in certain exceptional cases of rare occurrence, and of but little use in finding the longitude, see Art. 330). Then

$$\tau = \frac{hk \cos \psi}{n} - \frac{hm \cos (M - N)}{n}$$

or, when $\sin \psi$ is not very small,

$$\tau = \frac{hm}{n} \cdot \frac{\sin (M - N - \psi)}{\sin \psi}$$

If the local mean time t was observed, take $h = 3600$ in these formulæ, and then the longitude will be found by

$$\omega = T_0 - t + \tau$$

But if the local sidereal time μ was observed, take $h = 3609.856$ in the preceding formulæ; then, μ_0 being the sidereal time at the first meridian corresponding to T_0,

$$\omega = \mu_0 - \mu + \tau$$

The longitude thus found will be affected by the errors of the Ephemeris.

IV. To form the equations of condition for correcting the longitude for errors of the Ephemeris when the occultation has been observed at more than one place, compute the auxiliaries

$$T_1 = T_0 - \frac{1}{n} (x_0 \sin N + y_0 \cos N)$$
$$z = - x_0 \cos N + y_0 \sin N$$
$$\nu = \frac{h}{n \pi}$$

the same value of h being used as before.

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* According to OUDEMANS (*Astron. Nach.*, Vol. LI., p. 30), we should use for occultations $k = 0.27264$, or $\log k = 9.435590$, which amounts to taking the moon's apparent semidiameter about $1''.25$ greater in occultations than in solar eclipses. But it is only for the reduction of isolated observations that we need an exact value, since, when we have a number of observations, the correction of whatever value of k we may use will be obtained by the solution of our equations of condition.

1850 April 15.	a	δ	π
6^h	65° 56′ 21″.16	+ 16° 40′ 0″.05	58′ 55″.22
7	66 32 32 .06	16 46 30 .53	58 55 .87
8	67 8 46 .02	16 52 54 .77	58 56 .49
9	67 45 3 .02	16 59 12 .76	58 57 .10

The position of *Aldebaran* for the same date was

$$a' = 66° 49′ 33″.9 \qquad \delta' = + 16° 12′ 1″.7$$

Hence, by I. of the preceding article, we form the following table:

Gr. T.	x	x'	y	y'
6^h	− 0.86519	+ 0.58849	+ 0.47664	+ 0.10871
7	− 0.27671	47	.58531	63
8	+ 0.31176	42	.69390	56
9	+ 0.90014	32	.80243	48

II. The sidereal time of Greenwich Mean Noon, April 15, 1860, was $1^h 33^m 8^s.96$. With this number, converting the Königsberg times into mean times for the sake of uniformity, we find

	Cambridge.		Königsberg.	
	Immersion.	Emersion.	Immersion.	Emersion.
t	$2^h 1^m 52^s.45$	$3^h 1^m 38^s.35$	$9^h 23^m 15^s.64$	$10^h 13^m 11^s.88$
$t + \omega$	6 46 22.45	7 46 8 .35	8 1 15 .24	8 51 10.98
μ	54° 2′ 2″.55	69° 0′ 58″.35	164° 25′ 54″.90	176° 56′ 54″.00
$\mu - a'$	347 12 28 .65	2 11 24 .45	97 36 21 .00	101 7 20 .10
$\log \rho \sin \phi'$	9.826441		9.909898	
$\log \rho \cos \phi'$	9.869121		9.762639	
$\log \xi$	n9.214324	8.451362	9.758801	9.735287
$\log \eta$	9.646065	9.641159	9.904088	9.922175
$\log \zeta$	9.944427	9.952794	9.185091	8.549725

The value of $\log \zeta$ has been found in order to find the correction for refraction. This correction is here quite sensible in the case of the Königsberg observations which were made at a great

zenith distance. By the table on p. 517, we find that the logarithms of ξ and η must be increased by 0.000006 for immersion, and by 0.000041 for emersion. Applying these corrections, the values of the co-ordinates are as follows:

ξ	-0.16380	$+0.02827$	$+0.57386$	$+0.54366$
η	$+0.44266$	$+0.43768$	$+0.80175$	$+0.83602$

III. Assuming convenient times not far from $t + \omega$, we have

	$6^h.8$	$7^h.8$	$8^h.0$	$8^h.85$
Assumed T_0				
x_0	-0.39440	$+0.19406$	$+0.31176$	$+0.81188$
y_0	$+0.56358$	$+0.67218$	$+0.69890$	$+0.78615$
$x_0 - \xi = m \sin M$	-0.23460	$+0.16579$	-0.26210	$+0.26822$
$y_0 - \eta = m \cos M$	$+0.12092$	$+0.23450$	-0.10785	-0.04987
M	$297°\ 40'\ 16''.5$	$85°\ 15'\ 36''.1$	$247°\ 38'\ 1''.0$	$100°\ 31'\ 57''.7$
$\log m$	9.415608	9.458164	9.452433	9.435871
$x' = n \sin N$	$+0.58847$	$+0.58843$	$+0.58842$	$+0.58836$
$y' = n \cos N$	$+0.10865$	$+0.10857$	$+0.10856$	$+0.10849$
N	$79°\ 32'\ 21''.1$	$79°\ 32'\ 45''.8$	$79°\ 32'\ 48''.5$	$79°\ 33'\ 8''.5$
ψ	$216\ \ 11\ 35.9$	$312\ \ 33\ 59.0$	$167\ \ 35\ 28.5$	$21\ \ \ 1\ 28.1$
($h = 3600$) $\quad \tau$	$-89^s.74$	$-128^s.82$	$-68^s.63$	$-3^s.52$
ω	$4^h\ 44^m\ 37^s.81$	$4^h\ 44^m\ 12^s.83$	$-1^h\ 22^m\ 7^s.01$	$-1^h\ 22^m\ 4^s.90$

IV. For the equations of condition, taking $T_0 = 7^h.8$,

$$T_1 = \quad 7^h.2772 \qquad\qquad \pi = 3536''$$
$$\varkappa = +\ 0.6258 \qquad\qquad \log \nu = 0.2308$$

and putting

$$\omega_1 = \text{the true longitude of Cambridge,}$$
$$\omega_1' = \quad \text{``} \qquad \text{``} \qquad \text{Königsberg,}$$

we find, neglecting terms in Δee,

$$\omega_1 = \quad 4^h\ 44^m\ 37^s.81 - \nu\gamma + 1.245\ \vartheta - 2.108\ \pi\Delta k - 1.293\ \Delta\pi$$
$$\omega_1 = \quad 4\ \ 44\ \ 12.83 - \nu\gamma - 1.852\ \vartheta + 2.515\ \pi\Delta k + 1.660\ \Delta\pi$$
$$\omega_1' = -\ 1\ \ 22\ \ \ 7.01 - \nu\gamma - 0.374\ \vartheta - 1.742\ \pi\Delta k + 0.991\ \Delta\pi$$
$$\omega_1' = -\ 1\ \ 22\ \ 14.90 - \nu\gamma + 0.654\ \vartheta + 1.822\ \pi\Delta k + 1.195\ \Delta\pi$$

whence the two equations

$$0 = +\ 24^s.98 + 3.097\ \vartheta - 4.623\ \pi\Delta k - 2.953\ \Delta\pi$$
$$0 = +\ 7.89 - 1.028\ \vartheta - 3.564\ \pi\Delta k - 0.204\ \Delta\pi$$

If we determine ϑ and $\pi\Delta k$ in terms of $\Delta\pi$, these equations give

$$\vartheta = -\ 3''.33 + 0.607\ \Delta\pi$$
$$\pi\Delta k = +\ 3.17 - 0.232\ \Delta\pi$$

and then we find

$$\omega_1 = \quad 4^h\ 44^m\ 26^s.98 - \nu\gamma - 0.048\,\Delta\pi$$
$$\omega_1' = -1\quad 22\quad 11.29 - \nu\gamma + 1.169\,\Delta\pi$$

Assuming $\omega_1' = -1^h\ 22^m\ 0^s.4$ as well determined, the last equation gives

$$\nu\gamma = -10^s.89 + 1.169\,\Delta\pi$$

which substituted in the value of ω_1 gives

$$\omega_1 = 4^h\ 44^m\ 37^s.87 - 1.217\,\Delta\pi$$

Finally, adopting the correction of the parallax for this date as given in Mr. ADAMS's table (Appendix to the Nautical Almanac for 1856), $\Delta\pi = +5''.1$, this last value becomes

$$\omega_1 = 4^h\ 44^m\ 31^s.66$$

which agrees almost perfectly with the longitude of Cambridge found by the chronometric expeditions, which is $4^h\ 44^m\ 31^s.95$.

With the same value of $\Delta\pi$ we find

$$\gamma = -2''.90 \qquad \vartheta = -0''.23 \qquad \pi\Delta k = +1''.99$$

and hence, by (586), the corrections of the Ephemeris on this date, according to these observations, are

$$\Delta(\alpha - \alpha') = -2''.93 \qquad \Delta(\delta - \delta') = -0''.77$$

The value $\pi\Delta k = +1''.99$ gives $\Delta k = 0.00056$, and hence the corrected value $k = 0.27227 + 0.00056 = 0.27283$, which is not very different from OUDEMANS's value. (See p. 551).

342. When a number of occultations have been observed at a place for the determination of its longitude, it will usually be found that but few of the same occultations have been observed at other places. If, then, we were to depend altogether upon *corresponding* observations at other places, we should lose the greater part of our own. In order to employ all our data, we may in such case find for each date the corrections of the moon's place from meridian observations (see Art. 235), and, employing the corrected right ascension and declination in the computation of x and y, our equations of condition will involve only terms in $\pi\Delta k$ and $\Delta\pi$. The value of $\Delta\pi$ will, however, be different on

different dates, and, therefore, if we wish to retain this term, we must introduce in its stead the correction of the *mean* parallax which is the constant of parallax in the lunar tables. If this constant is denoted by π_0, we shall have, very nearly,

$$\Delta\pi = \frac{\pi}{\pi_0}\,\Delta\pi_0$$

where π is the parallax for the given date. The equations of condition will then be of the form

$$\omega_1 = \omega + a\,.\,\pi\Delta k + b\,.\,\Delta\pi_0$$

where

$$a = \nu \sec \psi \qquad\qquad b = \frac{\pi}{\pi_0}\,E$$

In HANSEN's Lunar Tables, now employed in the construction of our Ephemeris, $\pi_0 = 3422''.06$.

343. The passage of the moon through a well determined group of stars, such as the *Pleiades*, affords a peculiarly favorable opportunity for determining the correction of the moon's semi-diameter as well as of the moon's relative place, of the relative positions of the stars themselves, and also (if observations are made at distant but well determined places) of the parallax. Prof. PEIRCE has arranged the formulæ of computation, with a view to this special application, for the use of the U.S. Coast Survey. See Proceedings of the American Association for the Adv. of Science, 9th meeting, p. 97.

344. When an isolated observation of either an immersion or an emersion is to be computed, with no corresponding observations at other places, it will not be necessary to compute the values of x and y for a number of hours. It will be sufficient to compute them for the time $t + \omega$ (t being the observed local mean time, and ω the assumed longitude); and, as the correction of this time will always be small, the hourly changes may be found with sufficient precision by the approximate formulæ, easily deduced from (482),

$$x' = \frac{d\alpha}{\pi} \cos \delta \qquad\qquad y' = \frac{d\delta}{\pi}$$

where $d\alpha$ and $d\delta$ denote the hourly increase of α and δ respectively.

345. *To predict an occultation of a given star by the moon for a given place on the earth.*—We here suppose that it is already known that the star is to be occulted at the given place on a certain date, and that we wish to determine approximately the time of immersion and emersion in order to be prepared to observe it. The limiting parallels of latitude between which the occultation can be observed will be determined in the next article.

For a precise computation we proceed by Art. 322, making the modifications indicated in Art. 340.

But, for a sufficient approximation in preparing for the observation, the process may be abridged by assuming that the moon's right ascension and declination vary uniformly during the time of occultation, and neglecting the small variation of the parallax. It is then no longer necessary to compute the co-ordinates x and y directly for several different times at the first meridian, but only for any one assumed time, and then to deduce their values for any other time by means of their uniform changes. It will be most simple to find them for the time of true conjunction of the moon and star in right ascension, which is readily obtained by the aid of the hourly Ephemeris of the moon. Let this time be denoted by T_0. We have at this time $x = 0$, and the value of y will be found with sufficient accuracy by the formula

$$y_0 = \frac{\delta - \delta'}{\pi}$$

in which δ, π, are the moon's declination and horizontal parallax at the time T_0, and δ' is the star's declination.

Let $\Delta\alpha$ (in seconds of arc) and $\Delta\delta$ here denote the hourly changes of the moon's right ascension and declination for the time T_0. Then we have, nearly,

$$x' = \frac{\Delta\alpha}{\pi}\cos\delta \qquad y' = \frac{\Delta\delta}{\pi}$$

Let T_1 be any assumed time (which, in a first approximation, may be the time T_0 itself). Then the values of the co-ordinates at this time are

$$x = x'(T_1 - T_0) \qquad y = y_0 + y'(T_1 - T_0)$$

and to find the time (T) of contact of the star and the moon's limb, we shall, according to Art. 322, have the following formulæ:

$$\vartheta = \mu_1 - a' - \omega$$

in which μ_1 is the sidereal time at the first meridian corresponding to T_1, a' is the star's right ascension, and ω is the longitude ·

$$A \sin B = \rho \sin \varphi' \qquad\qquad \xi = \rho \cos \varphi' \sin \vartheta$$
$$A \cos B = \rho \cos \varphi' \cos \vartheta \qquad\qquad \eta = A \sin (B - \delta')$$

$$\mu' = 54148 \sin 1'' \qquad\qquad \xi' = \mu' A \cos B$$
$$\log \mu' = 9.41916 \qquad\qquad \eta' = \mu' \xi \sin \delta'$$

$$m \sin M = x - \xi \qquad\qquad n \sin N = x' - \xi'$$
$$m \cos M = y - \eta \qquad\qquad n \cos N = y' - \eta'$$

$$\sin \psi = \frac{m \sin (M - N)}{k} \qquad\qquad \log k = 9.43500$$

$$\tau = \frac{k \cos \psi}{n} - \frac{m \cos (M - N)}{n}$$

$$T = T_1 + \tau$$

where ψ is to be taken so that cos ψ shall be negative for immersion and positive for emersion.

For a second approximation, we take T as the assumed time T_1 and repeat the computation for immersion and emersion separately. The new value of ϑ for this second approximation will be most readily found by adding the sidereal equivalent of τ (converted into arc) to its former value.

It is more especially desirable to know the true time of emersion, and the angle of position of the point of reappearance of the star. Since this angle in solar eclipses was reckoned on the sun's limb, while here it must be reckoned on the moon's, it will be equal to $180 + Q$: so that, taking the value of ψ from the last approximation, we shall have

$$\left.\begin{array}{l}\text{Angle of pt. of contact from the}\\ \text{north pt. of the moon's limb}\end{array}\right\} = 180° + N + \psi$$

For the angle from the *vertex* of the moon's limb, we find γ by the equations

$$p \sin \gamma = \xi + \xi' \tau \qquad\qquad p \cos \gamma = \eta + \eta' \tau$$

where ξ, η, ξ', η', τ are to be taken from the last approximation; and then

$$\left. \begin{array}{l} \text{Angle of pt. of contact from} \\ \text{the vertex of the moon's limb} \end{array} \right\} = 180° + N + \psi - \gamma$$

If the computation in any case gives $m \sin (M - N) > k$, we have the impossible value $\sin \psi > 1$, which shows that the star is not occulted at the given place. If we wish to know how far the star is from the moon's limb at the time of nearest approach, we have (Art. 324)

$$\varDelta = \pm \, m \sin (M - N)$$

the sign being taken so that $\varDelta$ shall be positive. This is the linear distance of the moon's centre from the line drawn from the place of observation to the star, and therefore the angular distance as seen from the earth is $\pi \varDelta$. The apparent semidiameter of the moon is πk, and hence the apparent distance of the star from the moon's limb is $\pi (\varDelta - k)$.*

EXAMPLE.—Find the times of immersion and emersion in the occultation of *Aldebaran*, April 15, 1850, at Cambridge, Mass.

The elements of this occultation have been found on p. 553, with which an accurate computation may be made by the method of Art. 322; but, according to the preceding approximate method, we proceed as follows. The Greenwich time when the moon's right ascension was equal to that of the star is found, from the values of α on p. 553, to be

$$T_0 = 7^{\text{h}}.47 = 7^{\text{h}} \, 28^{\text{m}} \, 12^{\text{s}}.$$

For this time we have

$$\Delta\alpha = + 2174'' \qquad\qquad \delta = + 16° \, 49' \, 31''.1$$
$$\Delta\delta = + 384 \qquad\qquad\quad \delta' = 16 \quad 12 \quad 1 \,.7$$
$$\pi = 3536 \qquad\qquad \delta - \delta' = + 2249''$$

whence, by the above formulæ,

$$y_0 = + \, 0.6360 \qquad x' = + \, 0.5886 \qquad y' = + \, 0.1086$$

Then the computation for Cambridge, $\varphi = 42° \, 22' \, 49''$, $\omega = 4^{\text{h}} \, 44^{\text{m}} \, 30^{\text{s}}$, will be as follows. For the first approximation, we assume $T_1 = T_0$, and hence we have

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* More exactly, allowing for the augmentation of the moon's semidiameter, it is $\pi (\varDelta - k) (1 + \zeta \sin \pi)$, where we have $\zeta = A \cos (B - \delta')$.

$$T_1 = \quad 7^h\ 28^m\ 12^s.$$
$$\text{Sid. time Gr. noon} = \quad 1\ 33\quad 9\ .0$$
$$\text{Reduction for } T_1 = \quad \underline{\quad 1\ 13\ .6}$$
$$\mu_1 = \quad 9\ 2\ 34\ .6$$
$$\alpha' = \quad 4\ 27\ 18\ .3$$
$$\omega = \quad \underline{4\ 44\ 30}$$
$$\mu_1 - \alpha' - \omega = \vartheta = 23\ 50\ 46\ .3$$
$$= 357°\ 41'.6$$

with which we find the following results:

$x = \quad 0.$	$y = + 0.6360$
$\xi = -\ 0.0298$	$\eta = + 0.4377$
$m \sin M = + 0.0298$	$m \cos M = + 0.1983$
$M = \quad 8°\ 32'.4$	$\log m = \quad 9.3021$
$x' = + 0.5886$	$y' = + 0.1086$
$\xi' = + 0.1940$	$\eta' = -\ 0.0022$
$n \sin N = + 0.3946$	$n \cos N = + 0.1108$
$N = \quad 74°\ 19'.1$	$\log n = \quad 9.6127$
$\log \sin \psi = \quad n9.8395$	$\log \cos \psi = \quad 9.8590$
$-\dfrac{m \cos (M - N)}{n} = -\ 0^h.1690$	$\dfrac{k \cos \psi}{n} = \mp\ 0^h.4801$

For immersion.	For emersion.
$\tau = -\ 0^h.6491$	$\tau = +\ 0^h.3111$
$T_1 = \quad 7\ .4700$	$T_1 = \quad 7\ .4700$
$T = \quad 6\ .8209$	$T = \quad 7\ .7811$
$T = \quad 6^h\ 49^m\ 15^s$	$T = \quad 7^h\ 46^m\ 52^s$
$\omega = \quad 4\ 44\ 30$	$\omega = \quad 4\ 44\ 30$
Local time $= \quad 2\ 4\ 45$	Local time $= \quad 3\ 2\ 22$

These times are nearly correct enough; but, for a more accurate time of emersion, we now assume $T_1 = 7^h.7811$, with which we find

$$x = x'(T_1 - T_0) = + 0.1831 \qquad y'(T_1 - T_0) = + 0.0338$$
$$y_0 = + 0.6360$$
$$y = + 0.6698$$

and to find the new value of ϑ we have $\tau = +\ 0^h.3111 = 18^m\ 40^s$, the sidereal equivalent of which is $18^m\ 43^s.1$, or in arc $4°\ 40'.8$. This, added to the above value of ϑ, gives the corrected value $\vartheta = 2°\ 22'.4$. Repeating the computation with these new values of x, y, and ϑ, we find

$$-\frac{m \cos(M - N)}{n} = -0^{\text{h}}.5082 \qquad \begin{aligned} \psi &= 317^\circ \ 22' \\ N &= 74 \quad 55 \end{aligned}$$

$$\frac{k \cos \psi}{n} = + 0.4901 \qquad \frac{180}{212 \quad 17}$$

$$\begin{aligned} \tau &= -\overline{0.0181} \\ T_1 &= 7.7811 \end{aligned} \qquad \gamma = \frac{3 \quad 33}{208 \quad 34}$$

$$\begin{aligned} T &= \overline{7.7630} \\ &= 7^{\text{h}} 45^{\text{m}} 47^{\text{s}} \\ \text{Local time} &= 3 \quad 1 \quad 17 \end{aligned}$$

$\left. \begin{array}{l} \text{The star reappears at } 212^\circ \ 17' \\ \text{from the north point, or } 208^\circ \ 34' \\ \text{from the vertex, of the moon's} \\ \text{limb.} \end{array} \right\}$

This time agrees within 21ˢ with the actually observed time of emersion (given on p. 552). The principal part of the difference is due to the error of the Ephemeris on this date.

346. *To find the limiting parallels of latitude on the earth for a given occultation.*—The limiting curves within which the occultation of a given star is visible may be found by the general method given for solar eclipses, Art. 311, which, of course, may be much abridged in such an application. But, on account of the great number of stars which may be occulted, it is not possible to make even this abridged computation for all of them. The extreme parallels of latitude are, however, found by very simple formulæ, and may be used for each star.

For a point on the limiting curve, the least value of Δ in Art. 324 is in a solar eclipse $= L$, but in an occultation it is $= k$. Hence we have, by (557), the condition

$$\pm m \sin(M - N) = k$$

or, restoring the values of $m \sin M = x - \xi$, $m \cos M = y - \eta$,

$$(x - \xi) \cos N - (y - \eta) \sin N = \pm k$$

The angle N is here determined by the equations (552); but, for an approximate determination of the limits quite sufficient for our present purpose, we may neglect the changes of ξ and η, and take

$$n \sin N = x' \qquad n \cos N = y'$$

Let x_0 and y_0 be the values of x and y for the assumed epoch T_0; then for any time $T = T_0 + \tau$ we have

$$x = x_0 + n \sin N . \tau \qquad y = y_0 + n \cos N . \tau$$

which reduce the above condition to

$$(x_0 - \xi) \cos N - (y_0 - \eta) \sin N = \pm k$$

By the last equation of (500), we have, by neglecting the compression of the earth,

$$\sin \varphi = \eta \cos \delta' + \zeta \sin \delta'$$

in which

$$\zeta = \sqrt{(1 - \xi^2 - \eta^2)}$$

and we are now to determine the maximum and mininum values of φ, which fulfil these conditions.   Let us put

$$a = - \xi \cos N + \eta \sin N$$
$$b = \quad \xi \sin N + \eta \cos N$$

from which follow

$$\xi = - a \cos N + b \sin N$$
$$\eta = \quad a \sin N + b \cos N$$
$$\zeta = \quad \sqrt{(1 - a^2 - b^2)}$$

Then we also have, by our first condition,

$$a = - x_0 \cos N + y_0 \sin N \pm k$$

which is a constant quantity, since we may here assume x' and y' to be constant.

Since we have $a^2 + b^2 + \zeta^2 = 1$, we can assume γ and ϵ so as to satisfy the equations

$$\cos \gamma = a$$
$$\sin \gamma \cos \epsilon = b$$
$$\sin \gamma \sin \epsilon = \zeta$$

in which $\sin \gamma$ may be restricted to positive values.   The formula for φ thus becomes

$$\sin \varphi = \cos \gamma \sin N \cos \delta' + \sin \gamma \cos \epsilon \cos N \cos \delta' + \sin \gamma \sin \epsilon \sin \delta'$$

which may be put under a more simple form by assuming β and λ, so as to satisfy the conditions

$$\sin \beta = \sin N \cos \delta'$$
$$\cos \beta \cos \lambda = \cos N \cos \delta'$$
$$\cos \beta \sin \lambda = \sin \delta'$$

in which $\cos \beta$ may be restricted to positive values.

We thus obtain

$$\sin \varphi = \sin \beta \cos \gamma + \cos \beta \sin \gamma \cos (\lambda - \varepsilon)$$

in which φ and ε are the only variables. Since $\cos \beta \sin \gamma$ is positive, this value of $\sin \varphi$ is a maximum when $\cos (\lambda - \varepsilon) = 1$ or $\lambda - \varepsilon = 0$; and a minimum when $\cos (\lambda - \varepsilon) = -1$, or $\lambda - \varepsilon = 180°$. Hence we have, for the limits, $\sin \varphi = \sin (\beta \pm \gamma)$, that is

for the northern limit, $\varphi = \beta + \gamma$
for the southern limit, $\varphi = \beta - \gamma$

One of the points thus determined may, however, be upon that side of the earth which is farthest from the moon, since we have not restricted the sign of ζ, and our general equations express the condition that the point of observation lies in a line drawn from the star tangent to the moon's limb, which line intersects the surface of the earth in two points, for one of which ζ is positive and for the other negative. But, taking ζ only with the positive sign, we must also have $\sin \varepsilon$ positive. For the northern limit, therefore, when $\lambda = \varepsilon$, $\sin \lambda$ must be positive, which, according to the equation $\cos \beta \sin \lambda = \sin \delta'$, can be the case only when δ' is positive. Hence the formula $\varphi = \beta + \gamma$ gives the most northern limit of visibility only when the star is in north declination. For similar reasons, the formula $\varphi = \beta - \gamma$ gives the southern limit only when the star is in south declination. The second limit of visibility in each case must evidently be one of the points in which the general northern or southern limiting curve meets the rising and setting limits,—namely, the points where $\zeta = 0$, and consequently, also, $\sin \varepsilon = 0$, $\cos \varepsilon = \pm 1$, which conditions reduce the general formula for $\sin \varphi$ to the following :

$$\sin \varphi = (\sin N \cos \gamma \pm \cos N \sin \gamma) \cos \delta' = \sin (N \pm \gamma) \cos \delta'$$

If $\cos N$ is taken with the positive sign only, the upper sign in this equation will give the most northern limit to be used when the southern limit has been found by the formula $\varphi = \beta - \gamma$; and the lower sign will give the southern limit to be used when the northern limit has been found by the formula $\varphi = \beta + \gamma$.

Finally, since the epoch T_0 is arbitrary, we may assume for it the time of true conjunction in right ascension when $x_0 = 0$, and we shall then have

$$a = \cos \gamma = y_0 \sin N \pm k$$

The above discussion leads to the following simple arrangement of the formulæ

$$
\begin{aligned}
\cos \gamma_1 &= y_0 \sin N \pm 0.2723 & (\gamma < 180°) \\
\sin \beta &= \sin N \cos \delta' & (\beta < 90°) \\
\varphi_1 &= \beta \pm \gamma_1 \\
\cos \gamma_2 &= y_0 \sin N \mp 0.2723 \\
\sin \varphi_2 &= \sin (N \mp \gamma_2) \cos \delta' & (N < 90°)
\end{aligned} \Bigg\} \quad (604)
$$

in which the upper or the lower signs are to be used, according as the declination of the star is north or south. When the declination is north, φ_1 will be the northern limit and φ_2 the southern; and the reverse when the declination is south. The angle N is here supposed to be less than 90°, and is found by the formula

$$
\tan N = \frac{x'}{y'}
$$

always considering y' as well as x' to be positive.

When the cylindrical shadow extends beyond the earth, north or south, we shall obtain imaginary values for γ_1 or γ_2. The following obvious precepts must then be observed:

1st. When $\cos \gamma_1$ is imaginary, the occultation is visible beyond the pole which is elevated above the principal plane of reference, and, therefore, we must put for the extreme limit $\varphi_1 = + 90°$, or $\varphi_1 = - 90°$, according to the sign of δ'.

2d. When $\cos \gamma_2$ is imaginary, the value of φ_2 will be the latitude of that point of the (great circle) intersection of the principal plane and the earth's surface which lies nearest the depressed pole; that is, we must take $\varphi_2 = \delta' - 90°$, or $\varphi_2 = \delta' + 90°$, according as δ' is positive or negative.

It is also to be observed that the numerical value of φ_1 obtained by the formula $\varphi_1 = \beta \pm \gamma_1$ may exceed 90°, in which case the true value is either $\varphi_1 = 180° - (\beta \pm \gamma_1)$, or $\varphi_1 = - 180° - (\beta \pm \gamma_1)$, since these values have the same sine.

EXAMPLE.—Find the limiting parallels of latitude for the occultation of *Aldebaran*, April 15, 1850.

We have found, page 559, for this occultation,

$$
y_0 = + 0.6360 \qquad x' = 0.5886 \qquad y' = 0.1086
$$

Hence, with $\delta' = 16° 12'$, we find

$$N = \quad 79° 33'$$
$$y_0 \sin N = + 0.6255$$
$$k = \quad 0.2723$$
$$\cos \gamma_1 = + 0.8978$$
$$\cos \gamma_2 = + 0.3532$$
$$\gamma_2 = \quad 69° 19'$$
$$N - \gamma_2 = \quad 10 \quad 14$$

$$\log \sin \beta = 9.9751$$
$$\beta = 70° 47'$$
$$\gamma_1 = 26 \quad 8$$
$$\beta + \gamma_1 = 96 \quad 55$$
$$\varphi_1 = 83 \quad 5$$
$$\varphi_2 = \quad 9 \quad 49$$

It is hardly necessary to observe that the occultation is not visible at all the places included between the extreme latitudes thus found, since the true limiting curves do not coincide with the parallels of latitude, but cut the meridians at various angles, as is illustrated by the southern limit in our diagram of a solar eclipse, p. 504. Unless a place is considerably within the assigned limits, it may, therefore, be necessary in many cases to make a special computation, by the method of Art. 345, to determine whether the occultation can be observed.

OCCULTATIONS OF PLANETS BY THE MOON.

347. If the disc of a planet were always a circle, and fully illuminated, its occultation by the moon might be computed by the general method used for solar eclipses by merely substituting the parallax and semidiameter of the planet for those of the sun; and this is the method which has generally been prescribed by writers on this subject. But with the telescopes now in use, and especially with the aid of the electro-chronograph, it is possible to observe the instants of contact with the planet's limb to such a degree of accuracy that it appears to be worth while to take into account the true figure of the visible illuminated portion of the planet. Moreover, the investigation of this true figure possesses an intrinsic interest which justifies entering upon it here somewhat at length.

In order to embrace at once all cases, I shall consider the planet as a spheroidal body which even when fully illuminated presents an elliptical outline, and when partially illuminated presents an outline composed of two ellipses, of which one is the boundary of the spheroid and the other is the limit of illumination on the side of the planet towards the observer. I begin with the determination of the first of these ellipses.

348. *To find the apparent form of the disc of a spheroidal planet.**—
Let us first express the apparent place of any point of the
surface of the planet, by referring it to three planes perpen-
dicular to each other, of which the plane of xy coincides with the
plane of the planet's equator, while the axis of z coincides with
the axis of rotation. In this system, let

     $x, y, z =$ the co-ordinates of the point on the surface of the
               planet,
     $\xi, \eta, \zeta =$ those of the observer.

Straight lines drawn from the observer to the centre of the
planet and to the point on its surface determine their apparent
places on the celestial sphere. If these places are referred to
the great circle which corresponds to the planet's equator, and
if we put

     $\lambda, \lambda' =$ the geocentric longitudes of the apparent places of the
            planet's centre and the point on its surface, reckoned
            from the axis of x, in the great circle of the planet's
            equator,
     $\beta, \beta' =$ the latitudes of these places referred to the great
            circle of the planet's equator,
     $\rho, \rho' =$ the distances of the centre of the planet and the point
            on its surface from the observer,

we shall have (Arts. 32 and 33)†

$$\left. \begin{aligned} \rho \cos \beta \cos \lambda &= - \xi \\ \rho \cos \beta \sin \lambda &= - \eta \\ \rho \sin \beta &= - \zeta \end{aligned} \right\} \quad (605)$$

$$\left. \begin{aligned} \rho' \cos \beta' \cos \lambda' &= x - \xi \\ \rho' \cos \beta' \sin \lambda' &= y - \eta \\ \rho' \sin \beta' &= z - \zeta \end{aligned} \right\} \quad (606)$$

---

* The method of investigation here adopted, so far as relates to the apparent form of
the disc, is chiefly derived from BESSEL, *Astronomische Untersuchungen*, Vol. I. Art. VI.

† The group (606) may be deduced by supposing for a moment that the position
of the observer is referred to a system of planes parallel to the first, but having its
origin at the point on the surface of the planet. The co-ordinates in this system are
equal to those in the first increased respectively by x, y, and z. The negative sign
in the second members of both groups results from the consideration that the longi-
tude of the observer as seen from the planet is $180° + \lambda$, or $180° + \lambda'$; and his
latitude, $- \beta$, or $- \beta'$. Compare Art. 98.

Now, let O and C, Fig. 47, be the apparent places of the planet's centre and the point on its surface, projected upon the celestial sphere; Q the pole of the planet's equator; P the pole of the earth's equator; and let
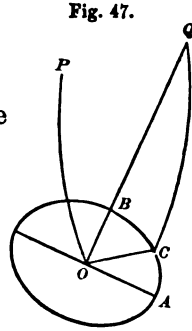
Fig. 47.

 $s' =$ the apparent distance of C from $O =$ the arc OC,

 $p' =$ the position angle of C reckoned at O, from the declination circle OP towards the east, $= POC$,

 $p =$ the position angle of the pole of the planet $= POQ$;

then, in the triangle QOC, we have

$$\sin s' \sin (p' - p) = \cos \beta' \sin (\lambda' - \lambda)$$
$$\sin s' \cos (p' - p) = \cos \beta \sin \beta' - \sin \beta \sin \beta' \cos (\lambda' - \lambda)$$

Multiplying these by ρ', and substituting the expressions (605) and (606), we obtain

$$\rho' \sin s' \sin (p' - p) = - x \sin \lambda + y \cos \lambda$$
$$\rho' \sin s' \cos (p' - p) = - x \sin \beta \cos \lambda - y \sin \beta \sin \lambda + z \cos \beta$$

or, since s' is very small and $\rho' \sin s'$ or $\rho' s'$ differs insensibly from $\rho \sin s'$ or $\rho s'$,

$$\left. \begin{array}{l} \rho s' \sin (p' - p) = - x \sin \lambda + y \cos \lambda \\ \rho s' \cos (p' - p) = - x \sin \beta \cos \lambda - y \sin \beta \sin \lambda + z \cos \beta \end{array} \right\} \quad (607)$$

These equations apply to any point on the surface of the planet. If we apply them to those points in which the visual line of the observer is *tangent* to that surface, they will determine the curve which forms the apparent disc. The equation of an ellipsoid of revolution whose axes are a and b, of which b is the axis of revolution, is

$$1 = \frac{xx}{aa} + \frac{yy}{aa} + \frac{zz}{bb} \qquad (608)$$

and the equation of a tangent line passing through the point whose co-ordinates are ξ, η, and ζ is

$$1 = \frac{x\xi}{aa} + \frac{y\eta}{aa} + \frac{z\zeta}{bb} \qquad (609)$$

The distances ξ, η, and ζ are very great in comparison with x,

y, and z. If we divide (609) by ρ, the quotients $\frac{\xi}{\rho}, \frac{\eta}{\rho}, \frac{\zeta}{\rho}$ will be of the same order as $\frac{x}{a}, \frac{y}{a}, \frac{z}{b}$, but the quotient $\frac{1}{\rho}$ will be inappreciable in relation to the quotients $\frac{x}{aa}, \frac{y}{aa}, \frac{z}{bb}$. Performing this division, therefore, and substituting the values of ξ, η, and ζ from (605), we may write for the equation of the tangent line

$$0 = \frac{x \cos \beta \cos \lambda}{aa} + \frac{y \cos \beta \sin \lambda}{aa} + \frac{z \sin \beta}{bb} \qquad (610)$$

If the curve ACB, Fig. 47, is referred to rectangular axes passing through the apparent centre O of the planet, one of which is in the direction of the pole of the planet, and if u and v denote the co-ordinates of any point of the curve, so that

$$u = s' \sin (p' - p)$$
$$v = s' \cos (p' - p)$$

the equations (607) and (610) will enable us to determine x, y, and z in terms of u and v. Putting

$$\frac{bb}{aa} = 1 - ee$$

the three equations become

$$\rho u = - x \sin \lambda + y \cos \lambda$$
$$\rho v = - (x \cos \lambda + y \sin \lambda) \sin \beta + z \cos \beta$$
$$0 = (x \cos \lambda + y \sin \lambda)(1 - ee) \cos \beta + z \sin \beta$$

from which we derive

$$- x \sin \lambda + y \cos \lambda = \rho u$$
$$- x \cos \lambda - y \sin \lambda = \rho v \, \frac{\sin \beta}{1 - ee \cos^2 \beta}$$
$$z = \rho v \, \frac{(1 - ee) \cos \beta}{1 - ee \cos^2 \beta}$$

Substituting these values in (608) and putting

$$s = \frac{a}{\rho} = \text{the greatest apparent semidiameter of the planet,}$$
$$c = \sqrt{(1 - ee \cos^2 \beta)}$$

we find

$$ss = uu + \frac{vv}{cc} \qquad (611)$$

which is the equation of the outline of the planet as projected upon the celestial sphere, or upon a plane passed through the centre of the planet at right angles to the line of vision. It represents an ellipse whose axes are $2s$ and $2s \sqrt{(1 - ee \cos^2 \beta)}$, e being the eccentricity of the planet's meridians. The minor axis (OB, Fig. 47) lies in the direction of the great circle drawn to the pole of the planet's equator.

We next proceed to determine what portion of this ellipse is illuminated and visible from the earth.

349. *To find the apparent curve of illumination of a planet's surface.*— If the sun be regarded as a point (which will produce no sensible error in this problem), the curve of illumination of the planet, *as seen from the sun*, can be determined by conditions quite similar to those employed in the preceding problem; for we have only to substitute the co-ordinates expressing the sun's position with reference to the planet, instead of those of the observer. If, therefore, we put

$A, B =$ the heliocentric longitude and latitude of the centre of the planet referred to the great circle of the planet's equator,

the equation of the tangent line from the sun to the planet, being of the same form as (610), will be

$$0 = \frac{x \cos B \cos A}{aa} + \frac{y \cos B \sin A}{aa} + \frac{z \sin B}{bb} \qquad (612)$$
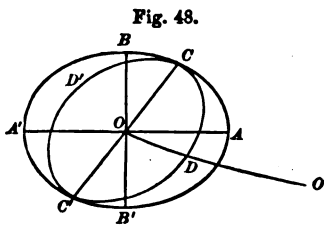
If each point which satisfies this condition be projected upon the celestial sphere by a line from the observer on the earth, and u and v again denote the co-ordinates of the projected curve, we have here, also, to satisfy the equations

$$\left.\begin{aligned} \rho u &= -x \sin \lambda + y \cos \lambda \\ \rho v &= -(x \cos \lambda + y \sin \lambda) \sin \beta + z \cos \beta \end{aligned}\right\} \quad (613)$$

in which λ and β have the same signification as in the preceding article. The values of x, y, and z, determined by the three equations (612), (613), being substituted in the equation of the ellipsoid, we obtain the relation between u and v, or the equation

which is the required equation of the curve of illumination, as seen from the earth, projected upon the celestial sphere. It represents an ellipse whose centre is at the origin but whose axes are, in general, inclined to the axes of co-ordinates, and, consequently, to the axes of the ellipse of equation (611). The equation (611) is only the particular case of (616) which corresponds to $V = 0$, or the case of full illumination.

350. We have yet to determine what portions of the apparent disc are bounded by the two curves

Fig. 48.

respectively. If $ABA'B'$, Fig. 48, is the ellipse of (611), which I shall call the *first ellipse*, and $CDC'D'$ that of (616), which I shall distinguish as the *second ellipse*, the visible outline of the planet is composed of one-half the first and one-half the second curve, and these halves either begin or end at the points C and C', which are the common points of tangency of the two curves. These points satisfy both equations; and, therefore, putting u_1 and v_1 for the co-ordinates of either point, and subtracting (611) from (616), we find

$$ 0 = \left(u_1 \sin w + v_1 \frac{\cos w}{c} \right)^2 \tan^2 V $$

which is satisfied, in general, by taking

$$ u_1 \sin w + v_1 \frac{\cos w}{c} = 0 $$

Denoting the position angle corresponding to $u_1 \, v_1$, by p_1, we have $u_1 = s_1 \sin (p_1 - p)$, $v_1 = s_1 \cos (p_1 - p)$. Substituting these values, and also putting

$$ c_1 \sin w_1 = \sin w \qquad\qquad c_1 \cos w_1 = \frac{\cos w}{c} \qquad (617) $$

the preceding condition becomes

$$ c_1 s_1 \cos (p_1 - p - w_1) = 0 $$

whence

$$ p_1 = p + w_1 \mp 90° \qquad (618) $$

which expresses the position angles of both C and C'. If we draw the arc ODO', Fig. 48, making the angle $BOO' = w$, and

take $OO' = V$, the point O' will be nearly the position of the planet as seen from the sun, and the arc V will be the measure of the angular distance between the sun and the earth as viewed from the planet. If we assume $\sin w$ to be positive in equations (615), as we are at liberty to do, the arc V will be reckoned from the planet *eastward* from 0° to 360°. Now, so long as V is less than 180°, the west limb will evidently be the full limb, and when V is greater than 180°, the east limb will be the full limb. Hence we infer that a point whose given position angle is p' is on the east limb when

$$p' > p + w_1 - 90° \quad \text{and} \quad < p + w_1 + 90°$$

but on the west limb when

$$p' < p + w_1 - 90° \quad \text{and} \quad > p + w_1 + 90°$$

When $V > 90°$ and $< 270°$, the planet is crescent; but when $V > 270°$ and $< 90°$, it is gibbous. In the case of a crescent planet there are two points, one on the full and the other on the crescent limb, corresponding to the same position angle: hence in observations of a crescent planet the point of observation on the limb will not be sufficiently determined by the position angle alone; it will be necessary for the observer to distinguish the crescent from the full limb in his record.

351. In order to apply the preceding theory, it is necessary to find the quantities p, β, λ, B, Λ. The direction of the axis of x in Art. 348 was left indeterminate, and may be assumed at pleasure, but it is most convenient to let it pass through the ascending node of the planet's equator on the equinoctial, so that λ and β will be reckoned from this node. The position of the node must, therefore, be known, and this we derive from the researches of physical astronomers. If we put

$n =$ the longitude of the ascending node of the planet's equator on the equinoctial,

$i =$ the inclination of the planet's equator to the equinoctial,

we have at any given time t, for the planets Jupiter and Saturn, the only ones whose figures are sensibly spheroidal,

we deduce from the data of BESSEL and DAMOISEAU, for a given year t,

For Jupiter.
$$\begin{cases} N' = 335° 40' 46'' + 49''.80 (t - 1850) \\ I' = 2° 8' 51'' + 0''.43 (t - 1850) \\ N = 336° 33' 18'' + 46''.55 (t - 1850) \end{cases}$$

For Saturn.
$$\begin{cases} N' = 167° 31' 52'' + 46''.62 (t - 1850) \\ I' = 28° 10' 27'' - 0''.35 (t - 1850) \\ N = 43° 31' 34'' - 86''.75 (t - 1850) - 0''.0625(t - 1850)^2 \end{cases}$$

and these values for Saturn also apply to the rings.

Finally, if we put

A', B' = the heliocentric longitude and latitude of the planet, referred to the ecliptic,

the formulæ (29) or (31) will serve to convert $A' - N'$ and B' into $A - N$ and B; and they become

$$\begin{aligned} K \sin M &= \tan B' & K' \sin (A - N) &= \cos (M - I') \\ K \cos M &= \sin (A' - N') & K' \cos (A - N) &= \cos M \cot (A' - N') \\ & \tan B = \sin (A - N) \tan (M - I') \end{aligned}$$ \qquad (620)

352. The preceding complete theory admits of several abridgments in its application to the different planets, varying according to the features peculiar to each.

Jupiter.—The inclination of Jupiter's equator to the ecliptic is so small that the quantity $c = \sqrt{(1 - ee \cos^2 \beta)}$ never differs sensibly from $\sqrt{(1 - ee)}$, which, according to STRUVE's measures, is 0.92723. I shall, therefore, use as a constant the value $\log c = 9.9672$. Again, on account of the small inclinations both of Jupiter's equator and of his orbit to the ecliptic, the angle w never differs much from 90°, and, since this angle is required only in computing the gibbosity of the planet (which never exceeds $0''.5$), it is plain that we may take $w = 90°$, and that V may be found with sufficient accuracy by the formula

$$V = A - \lambda$$

or, indeed, by the formula

$$V = A' - \lambda' \qquad (621)$$

in which A' and λ' are, respectively, the heliocentric and geocentric longitudes of the planet, the former being taken directly

sent the centres of the earth, the sun, and the planet; $S'O'O''$, the great circle of the celestial sphere whose plane passes through the three bodies; S' and O', the geocentric places of the sun and the planet; O'', the heliocentric place of the planet. Then $O'O''$ is the arc heretofore denoted by V, and, in the infinite sphere, is the measure of the angle $O'OO'' = SOE$. Putting then $V = O'O''$, $\gamma = S'O'$, and also
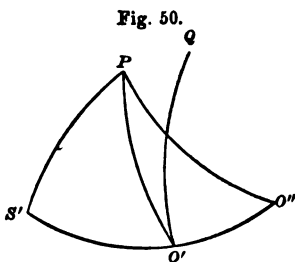
$$R' = SO = \text{the heliocentric distance of the planet,}$$
$$R = SE = \qquad `` \qquad `` \qquad `` \quad \text{earth,}$$

we have

$$\sin V = \frac{R}{R'} \sin \gamma$$

We might find V directly from the three known sides of the triangle SOE; but, as we have yet to find p, and γ comes out at the same time with p in a very simple manner, it will be preferable to employ the above form.

To find p and γ, let S', O', O'', Fig. 50, be the three places above referred to, and P the pole of the equinoctial. Draw $O'Q$ perpendicular to the great circle $S'O'O''$. This perpendicular passes through the adopted pole of the planet, and we have $PO'Q = p$, or $PO'S' = 90° - p$, and $S'O' = \gamma$. Hence, denoting by δ' and D the declination of the planet and the sun, and by α' and A their right ascensions respectively, the spherical triangle $PS'O'$ gives


Fig. 50.

$$
\left.
\begin{aligned}
\cos \gamma &= \sin \delta' \sin D + \cos \delta' \cos D \cos(\alpha' - A) \\
\sin \gamma \sin p &= \cos \delta' \sin D - \sin \delta' \cos D \cos(\alpha' - A) \\
\sin \gamma \cos p &= \cos D \sin(\alpha' - A)
\end{aligned}
\right\} \quad (622)
$$

Hence, introducing an auxiliary to facilitate the computation, both p and V will be found by the following formulæ:

$$
\left.
\begin{aligned}
\tan F &= \tan D \sec(\alpha' - A) \\
\tan p &= \cot(\alpha' - A) \sin(F - \delta') \sec F \\
\sin V &= \frac{R}{R'} \cdot \frac{\sin(\alpha' - A) \cos D}{\cos p}
\end{aligned}
\right\} \quad (623)
$$

In this method of finding V we do not determine whether it is

Let O, Fig. 51, be the apparent centre of the planet, and C the point of contact of its limb with that of the moon. Let OM be drawn from O towards the moon's centre, intersecting the moon's limb in D. Since the apparent semidiameter of any of the planets is never greater than $31''$, it is evident that no appreciable error can result from our assuming that the small portion CD of the moon's limb coincides sensibly with the common tangent to the two bodies drawn at C. If, then, the planet were a spherical body with the radius OD, the observed

Fig. 51.

time of contact would not be changed. We may, therefore, reduce the occultation of a planet to the general case of eclipse of one spherical body by another, by substituting the perpendicular OD for the radius of the disc of the eclipsed body. Let s'' denote this perpendicular; let OA and OQ be the axes of u and v respectively, to which the curve of illumination is referred by the equation (616); and let ϑ be the angle QOD which the perpendicular s'' makes with the axis of v. The equation of the tangent line CD referred to these axes is
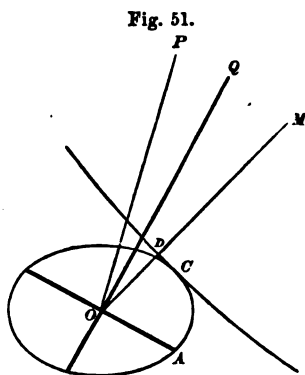
$$u \sin \vartheta + v \cos \vartheta = s'' \tag{625}$$

We have also in the curve

$$\frac{dv}{du} = - \tan \vartheta$$

Differentiating the equation (616), therefore, we have

$$\left(u \cos w - \frac{v \sin w}{c} \right)\left(\cos w + \frac{\tan \vartheta \sin w}{c} \right)$$

$$+ \left(u \sin w + \frac{v \cos w}{c} \right)\left(\sin w - \frac{\tan \vartheta \cos w}{c} \right) \sec^2 V = 0$$

By means of this equation, together with (616) and (625), we can eliminate u and v, and thus obtain the relation between s and s''. To abbreviate, put

$$x = u \cos w - \frac{v \sin w}{c}$$

$$y = u \sin w + \frac{v \cos w}{c}$$

and also

$$c' \sin \vartheta' = \frac{\sin \vartheta}{c} \qquad\qquad c' \cos \vartheta' = \cos \vartheta \qquad (626)$$

then the three equations become

$$x \cos (\vartheta' - w) - y \sin (\vartheta' - w) \sec^2 V = 0$$
$$x^2 + y^2 \sec^2 V = s^2$$
$$x \sin (\vartheta' - w) + y \cos (\vartheta' - w) = \frac{s''}{cc'}$$

From the first and second of these we find

$$x = \frac{s \sin (\vartheta' - w)}{\sqrt{[1 - \cos^2 (\vartheta' - w) \sin^2 V]}}$$
$$y = \frac{s \cos (\vartheta' - w) \cos^2 V}{\sqrt{[1 - \cos^2 (\vartheta' - w) \sin^2 V]}}$$

which substituted in the third give

$$s'' = s cc' \sqrt{[1 - \cos^2 (\vartheta' - w) \sin^2 V]}$$

Hence, if we put

we have

$$\left. \begin{array}{c} \sin \chi = \cos (\vartheta' - w) \sin V \\[2mm] s'' = s \cdot cc' \cos \chi \end{array} \right\} \quad (627)$$

We have seen (Art. 352) that in all practical cases we may take $w = 90°$, and, therefore, instead of (626) and (627) we may employ the following:

$$\left. \begin{array}{c} \tan \vartheta' = \dfrac{\tan \vartheta}{c} \\[3mm] \sin \chi = \sin \vartheta' \sin V \\[3mm] s'' = \dfrac{s \sin \vartheta \cos \chi}{\sin \vartheta'} \end{array} \right\} \quad (628)$$

If the occultation of a *cusp* of Venus or Mercury is observed, we have at once $s'' = s \cos \vartheta$ (for the axis of v coincides with the line joining the cusps), and we do not require V.

The value of s'' is to be substituted in (486) for the apparent semidiameter of the eclipsed body. In that formula, H denotes the apparent semidiameter at the distance unity: therefore, we must now substitute the value

$$\sin H = r' \sin s''$$

or, by (624) and (628),

$$\sin H = \frac{\sin s_0 \sin \vartheta \cos \chi}{\sin \vartheta'} \qquad (629)$$

Since f is here very small, we may put $\tan f = \sin f$, and the formula for L (488) becomes

$$L = (z - \zeta) \sin f \pm k$$
$$= (z - \zeta) \frac{\sin H}{r'g} \pm k \pm (z - \zeta) \frac{k \sin \pi_0}{r'g}$$

Hence, putting

$$k' = k + (z - \zeta) \frac{k \sin \pi_0}{r'g} \qquad (630)$$

we have

$$L = (z - \zeta) \frac{\sin H}{r'g} \pm k' \qquad (631)$$

When the angle ϑ is known, therefore, the preceding formulæ will determine L, with which the computation will be carried out in precisely the same form as in the case of a solar eclipse, Art. 329. To find ϑ, let OP, Fig. 51, be drawn in the direction of the pole of the equinoctial; then we have $POQ = p$, and, denoting POM by Q,

$$\vartheta = Q - p$$

and Q has here the same signification as in the general equations (567), as shown in Art. 295: so that when N and ψ have been found by (568) and (569), we have $Q = N + \psi$, or

$$\vartheta = N + \psi - p \qquad (632)$$

But to compute ψ by (569) we must know L, and this involves H, which depends upon ϑ. The problem can, therefore, be solved only by successive approximations; but this is a very slight objection in the present case, since the only formulæ to be repeated are those for L and ψ, and the second approximation will mostly be final. It can only be in a case such as the occultation of Saturn's ring, where the outline of the eclipsed body is very elliptical, and especially when the contact occurs near the northern or southern limb of the moon, that it may be necessary (for extreme accuracy) to compute H a second time and, consequently, ψ a third time.

The formula (629) is adapted to the general case of an ellip-

355. In the cases of the planets Neptune, Uranus, and the asteroids, the occultation of their centres will be observed, and it will be most convenient to compute by the method for a fixed star, only substituting for π the difference of the moon's and planet's horizontal parallaxes—that is, the *relative parallax*—in the formulæ for x and y, Art. 341.

This artifice of using the relative parallax may also be used with advantage for Jupiter and Saturn.

Having thus found x and y as for a fixed star, we shall have, in the preceding method,

$$L = (z - \zeta)\, \frac{\sin H}{r'} \pm k \qquad (634)$$

the other formulæ remaining unchanged.

EXAMPLE 1.—Several occultations of Saturn's Ring were observed by Dr. KANE at Van Rensslaer Harbor on the northwest coast of Greenland during the second Grinnell Expedition in search of Sir JOHN FRANKLIN.* The first of these was as follows:

1853 December 12th, Van Rensslaer Mean Time.
Immersion, contact of last point of ring, . . . $14^h\ 20^m\ 48^s.8$
Emersion,   "   "   "   "   . . . $14\ 54\ 18.3$

The assumed longitude of the place of observation was $\omega = 4^h\ 43^m\ 32^s$ west of Greenwich. The latitude was $\varphi = 78°\ 37'\ 4''$, whence

$$\log \rho \sin \varphi' = 9.989862 \qquad \log \rho \cos \varphi' = 9.296642$$

I. From the Nautical Almanac we take for 1853 Dec. 12, 19^h,

$$p = -\ 2°\ 37'.3 \qquad l = 24°\ 0'.4 \qquad \text{whence } \log c = \log \sin l = 9.6094$$

and from page 578, the outer ring only being observed,

$$s_0 = 187''.56 \qquad \log \sin s_0 = 6.9587$$

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* "Astronomical Observations in the Arctic Seas by ELISHA KENT KANE, M.D., U.S.N. Reduced and discussed by CHARLES A. SCHOTT, Assistant U.S. Coast Survey." Published by the Smithsonian Institution, May, 1860.

II. We shall compute the elements of the occultation for the centre of the planet for the Greenwich hours 18ʰ, 19ʰ, and 20ʰ. For these times we take the following quantities from the Nautical Almanac, applying to them the corrections determined by Mr. SCHOTT from the Greenwich observations of this date:

Moon.

Gr. T.	α	δ	π
18ʰ	3ʰ 36ᵐ 55ˢ.23	+ 18° 2' 47".5	54' 7".68
19	38 53 .92	12 13 .9	7 .22
20	40 52 .81	21 35 .7	6 .76

Saturn.

	α'	δ'	π'	log r'
18ʰ	3ʰ 39ᵐ 9ˢ.88	+ 17° 14' 28".4	1".05	0.9126
19	9 .16	26 .5		
20	8 .44	24 .5		

The corrections applied to the Nautical Almanac values to obtain the above are $\Delta\alpha = -\ 0^s.22$, $\Delta\delta = -\ 5''.0$, $\Delta\alpha' = +\ 0^s.15$, $\Delta\delta' = -\ 8''.9$, $\Delta\pi = +\ 0''.3$, this last correction being derived from Mr. ADAMS's Table in the Nautical Almanac for 1856.

We shall use the relative parallax, and compute as for a fixed star, taking $\pi - \pi'$ for π, namely

	π
18ʰ	54' 6".73
19	6 .17
20	5 .71

whence we find for the moon's co-ordinates,

Gr. T.	x	x'	y	y'
18ʰ	— 0.59152	+ 0.52457	+ 0.89382	+ 0.17436
19	— 0.06690	+ 0.52466	+ 1.06817	+ 0.17434
20	+ 0.45781	+ 0.52475	+ 1.24250	+ 0.17432

and, taking $z = r = \dfrac{1}{\sin \pi}$ for 19^h, as sufficiently accurate,

$$z = 63.54$$

III. For the co-ordinates of the place of observation:

	Immersion.	Emersion.
Local mean time t	$14^h\ 20^m\ 48^s.8$	$14^h\ 54^m\ 18^s.3$
$t + \omega$	$19\ \ 4\ \ 20\ .8$	$19\ \ 37\ \ 50\ .3$
Local sid. time μ	$117°\ 4'\ 59''.7$	$125°\ 28'\ 44''.7$

and hence, by the formulæ on p. 550,

	Immersion.	Emersion.
ξ	$+\ 0.17529$	$+\ 0.18685$
η	$+\ 0.90575$	$+\ 0.91363$
ζ	$+\ 0.38$	$+\ 0.35$
$z - \zeta$	63.16	63.19

IV. Assuming now two epochs corresponding nearly to the times of observation, the remainder of the computation *in extenso* is as follows:

	Immersion.	Emersion.
Assumed $T_0 \begin{cases} \\ \\ \end{cases}$	$19^h.07 =$ $19^h\ 4^m\ 12^s$	$19^h.63 =$ $19^h\ 37^m\ 48^s.$
x_0	$-\ 0.03017$	$+\ 0.26365$
y_0	$+\ 1.08037$	$+\ 1.17800$
$x_0 - \xi = m \sin M$	$-\ 0.20546$	$+\ 0.07680$
$y_0 - \eta = m \cos M$	$+\ 0.17462$	$+\ 0.26437$
M	$310°\ 21'\ 38''$	$16°\ 11'\ 56''$
$\log m$	9.43079	9.43980
$x' = n \sin N$	$+\ 0.52467$	$+\ 0.52472$
$y' = n \cos N$	$+\ 0.17434$	$+\ 0.17433$
N	$71°\ 37'\ 10''$	$71°\ 37'\ 20''$
$\log n$	9.74263	9.74266

Then, for a first appproximation, by the formula

$$\sin \psi = \frac{m \sin (M - N)}{\pm k}$$

and observing that the immersion is here an interior contact and the emersion an exterior contact, we have

	Immersion.	Emersion.
$h = 3600, \quad \log b = \log \dfrac{h\,L\,\cos \psi}{n}$	$n2.94455$	8.01171
$\log c = \log \dfrac{hm \cos (M - N)}{n}$	$n2.95956$	8.00741
b	$- 880^\cdot.1$	$+ 1027^\cdot.3$
c	$- 911 .1$	$+ 1017 .2$
$b - c = \tau$	$+ 31 .0$	$+ 10 .1$
Gr. Time of obs. $= T_0 + \tau = T$	$19^h \ 4^m \ 43^\cdot.0$	$19^h \ 37^m \ 58^\cdot.1$
$T - t = \omega$	$4 \ 43 \ 54 .2$	$4 \ 43 \ 39 .8$

If now we wish to form the equations of condition for deter-
mining the effect of errors in the data, we proceed precisely as
in the case of a solar eclipse, page 533, and find

	Immersion.	Emersion.
$\log \nu \tan \psi$	0.5341	$n0.4596$
$\log \nu \sec \psi$	0.5983	0.5454

where $\log \nu = \log \dfrac{3600}{n\pi} = 0.3023$. Hence, neglecting the terms

depending on the correction of the parallax and of the eccen-
tricity of the meridian, the equations of condition are

(Im.) $\omega_1 = 4^h 43^m 54^\cdot.2 - 2.001\,\gamma + 3.421\,\vartheta - 3.965\,\pi\Delta k$
(Em.) $\omega_1 = 4 \ 43 \ \ 39 .8 - 2.001\,\gamma - 2.881\,\vartheta + 3.511\,\pi\Delta k$

Eliminating ϑ from these equations, we have

$$\omega_1 = 4^h 43^m 46^\cdot.4 - 2.001\,\gamma + 0.092\,\pi\Delta k$$

An error of $1''$ in the moon's semidiameter (represented by $\pi\Delta k$)
would, therefore, have no sensible effect upon this combined
result; and since γ must also be very small, as we have corrected
the places of the moon and planet by the Greenwich observations,
we can adopt, as the definite result from this observation,

$$\omega_1 = 4^h 43^m 46^\cdot.4$$

It will be observed that in this example OUDEMANS's value,
$k = 0.27264$, has been employed; but our final equation shows
that the result would have been sensibly the same if we had
taken the usual value 0.27227; for the reduction of the result to
that which the latter value of k would have given is only
$0.092 \times 3247 \times (- 0.00037) = - 0^\cdot.11$.

EXAMPLE 2.—The occultation of *Venus*, April 24, 1860, was observed at the U. S. Military Academy, West Point ($\omega = 4^h 55^m 51^s$, $\varphi = 41° 23' 31''.2$), and at Albany ($\omega = 4^h 54^m 59^s.4$, $\varphi = 42° 39' 49''.5$), as follows:

	West Point. Sid. time.	Albany. Mean time.
Immersion.		
First contact, planet's full limb	$10^h 46^m 53^s.35$	$8^h 31^m 1^s.9$
Disappearance of cusp	10 47 47.80	8 31 54.2

The observations were made with the large refractors of the West Point and Dudley observatories.

I. To find p for the cusp observations, we have for the Greenwich time $13^h.478$, which is the mean of the times of the observations at the two places, and will serve for both,

$$\text{Planet, } a' = 78° 38'.6 \qquad \delta' = 25° 59'.1$$
$$\text{Sun, } \quad A = 32 \quad 45.5 \qquad D = 13 \quad 12.9$$

whence, by (623),

$$p = -7° 27'.3$$

and, from p. 578,

$$s_0 = 8''.55 \qquad \log \sin s_0 = 5.6175$$

II. We shall compute the moon's co-ordinates only for the Greenwich times $13^h.4$ and $13^h.5$. For these times the *American Ephemeris* furnishes the following data:

Moon.

Gr. T.	a	δ	π
$13^h.4$	$79° 12' 16''.8$	$+ 26° 43' 1''.6$	$57' 6''.6$
13 .5	79 15 58 .5	26 43 4 .3	57 6 .7

Venus.

	a'	δ'	$\log r'$
$13^h.4$	$78° 38' 23''.3$	$+ 25° 59' 2''.5$	9.9193
13 5	78 38 40 .7	25 59 4 .3	9.9193

Hence, by the formulæ of I. and II., p. 452, we find

	a	d	$\log g$
$13^h.4$	78° 38′ 17″.2	+ 25° 58′ 54″.5	9.9987
13 .5	78 38 34 .0	25 58 56 .3	9.9987

	x	x'	y	y'
13.4	+ 0.531695	+ 0.53390	+ 0.773681	+ 0.00480
13.5	+ 0.585085		+ 0.774161	

$$z = 60.19$$

III. For the co-ordinates of the places of observation:

	West Point.		Albany.	
	Full limb.	Cusp.	Full limb.	Cusp.
Local mean time t	$8^h 33^m 48^s.72$	$8^h 34^m 38^s.02$	$8^h 31^m 1^s.90$	$8^h 31^m 54^s.20$
$t + \omega$	13 29 34.72	13 30 29.02	13 26 1.30	13 26 53.60
μ	161° 43′ 20″.8	161° 56′ 57″.0	161° 2′ 44″.3	161° 15′ 50″.9
$\log \rho \sin \phi'$	9.818064		9.828792	
$\log \rho \cos \phi'$	9.875814		9.867157	
ξ	+ 0.745828	+ 0.746178	+ 0.780013	+ 0.780878
η	+ 0.551616	+ 0.552909	+ 0.568428	+ 0.564641
ζ	+ 0.87	+ 0.37	+ 0.38	+ 0.38
$z - \zeta$	59.82	59.82	59.81	59.81

IV. Assuming $T_0 = 13^h.45$, we find, for this time,

x_0	+ 0.558390			
y_0	+ 0.773921			
$x_0 - \xi = m \sin M$	− 0.187438	− 0.187788	− 0.171628	− 0.171988
$y_0 - \eta = m \cos M$	+ 0.222305	+ 0.221012	+ 0.210498	+ 0.209280
M	319° 51′ 50″	319° 38′ 47″	320° 48′ 30″	320° 35′ 11″
$\log m$	9.463563	9.462425	9.438915	9.432788
N	89° 29′ 6″			
$\log n$	9.727480			

Then, for the observations of the full limb, we have for both places, by (631), putting $H = s_0$,

	$\log (z - \zeta)$	1.7768	. . . . . .	1.7768
	ar. co. $\log r'g$	0.0820	. . . . . .	0.0820
$k = 0.27264$	constant	5.0542	$\log \sin s_0$	5.6175
0.00082 . . . . log		6.9130		
$k' = 0.27346$				
0.00299 . . . . log (1)				7.4768
$L = 0.27645$				

sun, p. 533. For the full limb we have only to substitute Δs_0 for ΔH; but for the cusp we must evidently substitute $\Delta s_0 \cos \vartheta$ for ΔH. It will be more accurate to restore $r'g$ in the place of r', since g here differs sensibly from unity. We shall thus find

$$\omega' = 4^h 55^m 54^s.0 - 1.967\,\gamma + 2.720\,\vartheta - 3.358\,\pi\Delta k - 4.061\,\Delta s_0$$
$$\omega' = 4\ \ 55\ \ \ 55.7 - 1.967\,\gamma + 2.844\,\vartheta - 3.459\,\pi\Delta k + 3.697\,\Delta s_0$$
$$\omega'' = 4\ \ 55\ \ \ \ 5.4 - 1.967\,\gamma + 2.352\,\vartheta - 3.067\,\pi\Delta k - 3.704\,\Delta s_0$$
$$\omega'' = 4\ \ 55\ \ \ \ 5.4 - 1.967\,\gamma + 2.438\,\vartheta - 3.134\,\pi\Delta k + 3.349\,\Delta s_0$$

where ω' and ω'' denote the true longitudes. Hence, also,

$$\omega' - \omega'' = + 48^s.6 + 0.368\,\vartheta - 0.291\,\pi\Delta k - 0.357\,\Delta s_0$$
$$\omega' - \omega'' = + 50.3 + 0.406\,\vartheta - 0.325\,\pi\Delta k + 0.348\,\Delta s_0$$

and the mean is

$$\omega' - \omega'' = + 49^s.5 + 0.387\,\vartheta - 0.308\,\pi\Delta k - 0.005\,\Delta s_0$$

The effect of an error in s_0 upon the difference of longitude of the two places is, therefore, insensible; but, to eliminate ϑ and $\pi\Delta k$, observations of the emersion should also be used. The effect of γ and ϑ upon ω' and ω'' can only be eliminated by means of observations of the moon's place at a standard observatory on the day of the observation, as we have already shown in other examples.

TRANSITS OF VENUS AND MERCURY.

856. The transits of Venus and Mercury may be computed by the method for solar eclipses, substituting the planet for the moon. In the formulæ (486), (487), &c., we must employ

$$\text{for Venus,} \quad k = 0.9975$$
$$\text{for Mercury,} \quad k = 0.3897$$

which are the values which result from the apparent semi-diameters of these planets adopted on p. 578.

Since b is no longer a small quantity, it will be necessary to employ the exact formulæ (479) instead of (481).

The longitude of a place at which the transit is observed may be computed from each of the four contacts of the limb of the sun and planet, by the formulæ of Art. 329. These observations, however, are of little use in determining an unknown longitude, on account of the great effect of small errors in the assumed
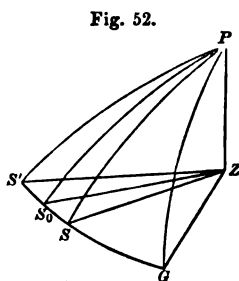
The discussion of all the equations of condition of the form (635) will, therefore, give not only the correction $\Delta\pi$ of the planet's parallax, but also, by the last-mentioned relation, that of the solar parallax.*

The transits of Venus will afford a far more accurate determination of this parallax than those of Mercury; for, on account of its greater proximity to the earth, the difference in the duration of the transit at different places will be much greater, and the coefficient of $\Delta\pi$ in the final equations proportionally great.

Although the general method for eclipses may also be extended to the prediction of the transits of the planets (by Art. 322), yet it is more convenient in practice to follow a special method in which advantage is taken of the circumstance that the parallaxes of both bodies are so small that their squares and higher powers may be neglected. LAGRANGE's method for this purpose is the most simple, and, in the improved form which I shall give to it in the following article, most accurate.

357. *To predict the times of ingress and egress for a given place.*— We first find the times of ingress and egress *for the centre of the earth*, from which the times for any place on the surface are readily deduced.

Let α, δ, α', δ' be the right ascensions and declinations of the planet and the sun for an assumed time T_0, at the first meridian, near the time of conjunction. Let m denote the apparent distance of the centres at this time. Let S' and S, Fig. 52, be the geocentric places of the centres of the sun and planet, P the pole; then, denoting the angles $PS'S$ and PSS' by P' and $180° - P$, the triangle PSS' gives

Fig. 52.

$$\sin \tfrac{1}{2} m \sin \tfrac{1}{2}(P + P') = \sin \tfrac{1}{2}(\alpha - \alpha') \cos \tfrac{1}{2}(\delta + \delta')$$
$$\sin \tfrac{1}{2} m \cos \tfrac{1}{2}(P + P') = \cos \tfrac{1}{2}(\alpha - \alpha') \sin \tfrac{1}{2}(\delta - \delta')$$

But, since $\tfrac{1}{2} m$ is at the time of a contact only about 8', we may without appreciable error substitute it for its sine, and,

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* Another method of forming the equations, apparently shorter, but in reality, where many observations are to be reduced, not more convenient than the rigorous method, will be found in ENCKE's *Die Entfernung der Sonne von der Erde, aus der Venusdurchgange von 1761 hergeleitet;* and *Der Venusdurchgange von 1769.*

The formulæ (636), (637), and (638) serve for the complete prediction for the centre of the earth.

To find the time of a contact for any point of the surface of the earth, let m be the geocentric apparent distance of the centres of the two bodies at any given time; m' the apparent distance, at the same time, as seen from a point on the earth's surface in latitude φ and longitude ω; π and π' the equatorial horizontal parallaxes of the planet and sun respectively; ζ and ζ' their geocentric zenith distances; ρ the radius of the earth for the latitude φ. The apparent zenith distances are $\zeta + \rho \pi \sin \zeta$ and $\zeta' + \rho \pi' \sin \zeta'$: these approximations being quite exact where the parallaxes are so small. Let Z, Fig. 52, be the geocentric zenith of the place, S and S' the true places of the bodies. The distance $SS' = m$ will become the apparent distance m' if we increase the sides ZS and ZS' by $\rho \pi \sin \zeta$ and $\rho \pi' \sin \zeta'$; and, if we regard these small increments as differentials, we shall have, by the first equation of (46),

$$m' - m = - \rho \pi \sin \zeta \cos S + \rho \pi' \sin \zeta' \cos S'$$

where $S = 180° - ZSS'$, and $S' = ZS'S$.

Let S_0 be the middle point of the arc SS', and denote the angle ZS_0S by S_0, the arc ZS_0 by ζ_0; then we have

$$- \sin \zeta \cos S = \sin \tfrac{1}{2} m \cos \zeta_0 - \cos \tfrac{1}{2} m \sin \zeta_0 \cos S_0$$
$$\sin \zeta' \cos S' = \sin \tfrac{1}{2} m \cos \zeta_0 + \cos \tfrac{1}{2} m \sin \zeta_0 \cos S_0$$

which give

$$m' - m = \rho \left[(\pi + \pi') \sin \tfrac{1}{2} m \cos \zeta_0 - (\pi - \pi') \cos \tfrac{1}{2} m \sin \zeta_0 \cos S_0 \right]$$

If then g and γ are determined by the conditions

$$\left. \begin{array}{l} g \sin \gamma = (\pi + \pi') \sin \tfrac{1}{2} m \\ g \cos \gamma = (\pi - \pi') \cos \tfrac{1}{2} m \end{array} \right\} \quad (639)$$

we have

$$m' - m = g\rho (\sin \gamma \cos \zeta_0 - \cos \gamma \sin \zeta_0 \cos S_0)$$

Produce the arc $S'S$, and take $S_0 G = 90° + \gamma$. Then, denoting the arc ZG by λ, the triangle ZGS_0 gives

$$\cos \lambda = - \sin \gamma \cos \zeta_0 + \cos \gamma \sin \zeta_0 \cos S_0$$

and the expression for m' becomes

$$m' = m - g\rho \cos \lambda \qquad (640)$$

In order to apply these formulæ in predicting the time of a contact at a given place, we observe, first, that this time differs but a few minutes from the time of the same contact for the centre of the earth, and during these few minutes we may assume the distance m to vary uniformly.

Let T be the time of the geocentric contact, and T' the required time of the contact at the place, both times being reckoned at the first meridian. At the time T the geocentric distance $= s' \pm s$, and at the time T' the apparent distance $m' = s' \pm s$ (neglecting here the augmentation of the semidiameters, which are too minute to be considered in merely predicting the phenomenon); but at this time T' the geocentric distance has become

$$m = s' \pm s + (T' - T) \frac{dm}{dt}$$

where $\frac{dm}{dt}$ denotes the change of m in the unit of time. These values substituted in (640) give

$$(T' - T) \frac{dm}{dt} = g\rho \cos \lambda$$

Differentiating (636), we find

$$\frac{dm}{dt} \sin M + \frac{dM}{dt} m \cos M = a \cos \delta_0 = n \sin N$$

$$\frac{dm}{dt} \cos M - \frac{dM}{dt} m \sin M = d \qquad = n \cos N$$

whence

$$\frac{dm}{dt} = n \cos (M - N)$$

But, since at the time T we have $m = s' \pm s$, we also have for this time, by (638), $M - N = \psi$, and, therefore,

$$\frac{dm}{dt} = n \cos \psi$$

which gives

$$T' = T + \frac{g\rho \cos \lambda}{n \cos \psi} \tag{644}$$

in which the values of n and ψ found in the computation for the centre of the earth are to be employed. The value of λ to be employed must be that which results from the preceding formulæ at the time T. Now, at this time the value of the angle

M is Q, which is found by (638), and this value is to be employed in (642), while in (641) we take $m = s' \pm s$.

The formula for T' will be

$$T' = T + \frac{\pi - \pi'}{n \cos \psi} \left[\rho \sin \varphi' \sin D + \rho \cos \varphi' \cos D \cos (\Theta - \omega) \right] \quad (645)$$

in which T, n, ψ, D, Θ, $\pi - \pi'$ are all constants, found in the computation for the centre: so that the computation for a particular place requires only this single formula in which the latitude and longitude of the place are to be substituted.

358. The necessary formulæ for the complete prediction are recapitulated as follows:

I.—FOR THE CENTRE OF THE EARTH.

Assume a convenient time T_0 near the time of true conjunction of the sun and the planet, or this time itself, reckoned at the first meridian, and find for this time the values of α, δ for the planet; α', δ' for the sun; the semidiameters s and s'; and the relative changes in right ascension and declination, a and d, in the unit of time. Then, putting $\delta_0 = \frac{1}{2}(\delta + \delta')$, compute

$$
\begin{aligned}
m \sin M &= (\alpha - \alpha') \cos \delta_0 & n \sin N &= a \cos \delta_0 \\
m \cos M &= \delta - \delta' & n \cos N &= d
\end{aligned}
$$

$$\sin \psi = \frac{m \sin (M - N)}{s' \pm s}$$

where $s' + s$ is to be employed for exterior contact, and $s' - s$ for interior contact. Putting $h = 3600$, to reduce the terms to seconds, we then find

$$T = T_0 - h \left(\frac{s' \pm s}{n} \right) \cos \psi - \frac{hm}{n} \cos (M - N)$$

in which $\cos \psi$ is to be taken with the negative sign for ingress and with the positive sign for egress.

For the greatest precision, the computation may be repeated separately for ingress and egress, taking for T_0 the value of T first computed.

As in solar eclipses, if T_1 denotes the time of nearest approach of the centres of the bodies, and $\varDelta_1$ the distance at this time, we have

$$\varDelta_1 = m \sin (M - N) \qquad\qquad T_1 = T_0 - \frac{hm}{n} \cos (M - N)$$

II.—CONSTANTS.

For each of the computed values of T take the corresponding values of N and ψ from the preceding computation. Then

$$Q = N + \psi$$

Take the horizontal parallaxes π and π' of the planet and the sun, and compute A and D by the formulæ

$$\gamma = \frac{\pi + \pi'}{\pi - \pi} \cdot \tan \tfrac{1}{2}(s' \pm s) \qquad \begin{aligned} f \sin F &= \sin \gamma \\ f \cos F &= \cos \gamma \cos Q \end{aligned}$$

$$\begin{aligned} \cos D \sin (A - \alpha_0) &= \quad \cos \gamma \sin Q \\ \cos D \cos (A - \alpha_0) &= -f \sin (\delta_0 + F) \\ \sin D &= \quad f \cos (\delta_0 + F) \end{aligned}$$

in which α_0 is the mean of the right ascensions of the planet and sun, and δ_0 the mean of their declinations, at the time T.

Find the sidereal time μ at the first meridian corresponding to T. Then form the three constants

$$\Theta = \mu - A \qquad B = \frac{\pi - \pi'}{n \cos \psi} h \sin D \qquad C = \frac{\pi - \pi'}{n \cos \psi} h \cos D$$

III.—FOR A GIVEN PLACE WHOSE LATITUDE IS φ AND WEST LONGITUDE ω.

Find the values of $\rho \sin \varphi'$ and $\rho \cos \varphi'$ by the geodetic table. The required time of the phenomenon at the place is

$$T' = T + B \cdot \rho \sin \varphi' + C \cdot \rho \cos \varphi' \cos (\Theta - \omega)$$

The local time will be $T' - \omega$. The angle Q will express the angular distance of the point of contact reckoned on the sun's limb from its north point towards the east, and will be very nearly the same for all places on the earth.

EXAMPLE.—Compute the times of ingress and egress for the transit of *Mercury*, November 11, 1861.

I. *For the centre of the earth.*—Let us take as the first meridian that of Washington, and employ the elements given in the American Ephemeris.

The Washington mean time of conjunction in right ascension is November 11, $14^h 59^m 43^s.6$, which we shall adopt as the value of T_0. For this time we have

III. For any place on the surface of the earth we have, there-fore, in mean Washington time,

Ingress, $T'' = 12^h\ 8^m\ 47^s.8 - 16^s.66\ \rho \sin \phi' - 52^s.19\ \rho \cos \phi' \cos (101°\ 1'.6 - \omega)$
Egress,　$T'' = 16\ 12\ 23.8 + 48.10\ \rho \sin \phi' + 26.23\ \rho \cos \phi' \cos (303\ 20.5 - \omega)$

or, in a more convenient form, giving the logarithms of the constant factors,

Ingress, $T'' = 12^h\ 8^m\ 47^s.8 - [1.2217]\ \rho \sin \phi' + [1.7176]\ \rho \cos \phi' \cos (\omega + 78°\ 58'.4)$
Egress,　$T'' = 16\ 12\ 23.8 + [1.6821]\ \rho \sin \phi' + [1.4187]\ \rho \cos \phi' \cos (\omega + 56\ 39.5)$

To determine whether the phenomenon is visible at the given place, we have only to determine whether the sun is above the horizon at the computed time. All the places at which it will be visible will be readily found by the aid of an artificial terres-trial globe, by taking that point where the sun is in the zenith at the time T, and describing a great circle from this point as a pole. All places within the hemisphere containing this pole evidently have the sun above the horizon. In the present example this point at ingress is in latitude $- 17°\ 43'$ and longi-tude $186°\ 2'$ west from Washington; and at egress it is in lati-tude $- 17°\ 46'$ and longitude $247°\ 4'$. The whole transit is invisible in the United States, and in Europe only the egress is visible.

For the egress at *Altona*, $\varphi = 53°\ 32'.8$, $\omega = 350°\ 3'.5$, we find

$$T' = \quad\ 16^h\ 13^m\ 13^s.0$$
$$\omega = -\ \ 5\ 47\ 57.4$$
$$\text{Altona mean time of egress} = \quad 22\ \ 1\ 10.4$$

The time actually observed by PETERSEN and PAPE was $22^h\ 1^m\ 8^s.5$.* The error of the prediction is very small, and proves the excellence of LE VERRIER's Tables of Mercury, from which the places in the American Ephemeris were derived.

OCCULTATION OF A FIXED STAR BY A PLANET.

359. Very small stars disappear to the eye when near the bright limb of a planet, before they are actually occulted by it; and the occultations of stars of sufficient brightness to be ob-served at the limb of the planet are so rare that it has not been thought worth while to incur the labor of predicting their oc-

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* *Astron. Nach.*, Vol. LVI. p. 239.

from the changes in the position of the plane of the orbit and the plane of the equator.

361. The variations of astronomical elements are usually divided into *secular* and *periodic*.

Secular variations are very slow changes, which proceed through ages (*secula*), so that for a number of years, or even centuries in some cases, they are nearly proportional to the time.

Periodic variations are relatively quick changes, which oscillate between their extreme values in so short a period that they cannot be regarded as proportional to the time except for very small intervals.*

The *true position* of a celestial body, or of a celestial plane, at a given time, is that which it actually has at that time; its *mean position* is that which it would have at that time if it were freed from its periodic variations.

362. The plane of the ecliptic, or of the earth's orbit, is a slowly moving plane. Its position at any epoch, as the beginning of the year 1800, can be adopted as a *fixed plane*, to which its position at any other time may be referred.

The plane of the equator is also a moving plane. Its inclination to the fixed plane and the direction of the line in which it intersects that plane are constantly changing, thus causing variations in the obliquity of the ecliptic and in the position of the equinoctial points.

363. The latitudes and declinations of stars are therefore subject to variations which do not arise from the motions of the stars, but from the shifting of the planes of reference; and the longitudes and right ascensions are in like manner subject to variations from the shifting of the vernal equinox, which is their common point of reference, or origin, from which both are reckoned.

Under the head of *precession* are considered those parts of these variations which are *secular;* namely, those which arise from the motions of the *mean ecliptic* and the *mean equator.*

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* Most of the secular variations also have periods, though of great length, and therefore not yet in all cases well defined: so that, strictly speaking, the distinction between secular and periodic variations is only an arbitrary one, established for practical convenience between variations of *long* and *short* periods.

changes their right ascensions, their longitudes, and their latitudes (Art. 23).

367. *Obliquity of the ecliptic.*—Since by the mutual action of the planets the position of the plane of the (mean) ecliptic is changed while that of the equator remains fixed, the mutual inclination of these planes, or the obliquity of the ecliptic, is changed.

The action of the sun and moon in causing luni-solar precession does not directly produce any change in the obliquity of the ecliptic; but, in consequence of the change produced by the planets, the attraction of the sun and moon is modified: so that there results an additional very minute change of the inclination of the mean equator to the fixed plane of reference.

These changes produce small changes in the co-ordinates of the stars, which, being secular in their character, are combined with the preceding in deducing the general precession.

368. *To find the general precession in longitude, and the position of the mean ecliptic, at a given time.*—Let NL, Fig. 53, be the fixed ecliptic, or the mean ecliptic at the beginning of the year 1800; AQ, the mean equator, and V the mean vernal equinox, or, as it is briefly called, the mean equinox, of 1800. In the figure, let the longitudes be reckoned from V towards N. Let

Fig. 53.
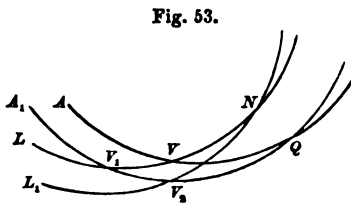
VV_1 be the luni-solar precession in longitude in the time t, and A_1Q the mean equator at the time $1800 + t$. By the action of the planets, the ecliptic in the same time is moved into the position NL_1: so that V_1V_2 is the planetary precession in the time t, and V_2 is the mean equinox at the time $1800 + t$.

The point N may be called the *ascending node* of the mean ecliptic on the fixed ecliptic.

The difference between NV and NV_2 is called the *general precession in longitude*, being that part of the change of the longitudes of the stars which is common to all of them.

Now, let us put

ϵ_0 = the mean obliquity of the ecliptic for 1800,

= NVQ,

ϵ_1 = the obliquity of the fixed ecliptic at the time $1800 + t$,

= NV_1Q,

ϵ = the mean obliquity of the ecliptic at the time $1800 + t$,
  = $NV_2 Q$,

ϑ = the planetary precession in the interval t,
  = $V_1 V_2$,

ψ = the luni-solar precession in the interval t,
  = VV_1,

ψ_1 = the general precession in the interval t,
  = $NV_2 - NV$,

Π = the longitude of the ascending node of the mean ecliptic at the time $1800 + t$, reckoned on the fixed ecliptic from the mean equinox of 1800,
  = VN,

π = the inclination of the mean ecliptic to the fixed ecliptic at the time $1800 + t$,
  = $V_1 N V_2$.

The first five of these quantities will be here assumed as known from the investigations of physical astronomers. The following are their values, according to STRUVE and PETERS,* for the epoch 1800:

$$
\left.
\begin{aligned}
\epsilon_0 &= 23° \ 27' \ 54''.22 \\
\epsilon_1 &= \epsilon_0 + 0''.00000735\,t^2 \\
\epsilon &= \epsilon_0 - 0''.4738\,t - 0''.0000014\,t^2 \\
\vartheta &= 0''.15119\,t - 0''.00024186\,t^2 \\
\psi &= 50''.3798\,t - 0''.0001084\,t^2
\end{aligned}
\right\} \quad (646)
$$

from which we can find ψ_1, Π, and π, as follows. In the triangle $NV_1 V_2$ we have

$$\pi = V_1 N V_2 \qquad\qquad \vartheta = V_1 V_2$$
$$180° - \epsilon = NV_2 V_1 \qquad \Pi + \psi = NV_1$$
$$\epsilon_1 = NV_1 V_2 \qquad\qquad \Pi + \psi_1 = NV_2$$

and hence, by the GAUSSIAN equations [Sph. Trig. (44)]

---

* DR. C. A. F. PETERS, *Numerus Constans Nutationis*, pp. 66 et 71. The observations at Dorpat give $0''.4645$ for the annual diminution of the obliquity, and this is adopted in the American Ephemeris instead of $0''.4738$, which results from theory and is subject to an error in the estimated mass of Venus. The difference, however, is so small that either number will serve to represent the actually observed obliquity for half a century within $0''.5$.

I have here adopted the precession constant ($50''.3798$) given by PETERS, rather for the convenience of the reader (this being employed in the English and American Almanacs) than on account of its superior accuracy. Recent researches rather confirm BESSEL's constant ($50''.36354$). See MÄDLER's *Die Eigenbewegungen der Fixsterne*, Dorpat, 1856, p. 11.

$$\left.\begin{array}{l} \cos \tfrac{1}{2}\pi \sin \tfrac{1}{2}(\psi - \psi_1) = \sin \tfrac{1}{2}\vartheta \cos \tfrac{1}{2}(\varepsilon + \varepsilon_1) \\ \cos \tfrac{1}{2}\pi \cos \tfrac{1}{2}(\psi - \psi_1) = \cos \tfrac{1}{2}\vartheta \cos \tfrac{1}{2}(\varepsilon - \varepsilon_1) \\ \sin \tfrac{1}{2}\pi \sin (\Pi + \tfrac{1}{2}\psi + \tfrac{1}{2}\psi_1) = \sin \tfrac{1}{2}\vartheta \sin \tfrac{1}{2}(\varepsilon + \varepsilon_1) \\ \sin \tfrac{1}{2}\pi \cos (\Pi + \tfrac{1}{2}\psi + \tfrac{1}{2}\psi_1) = \cos \tfrac{1}{2}\vartheta \sin \tfrac{1}{2}(\varepsilon - \varepsilon_1) \end{array}\right\} \quad (647)$$

The angles $\tfrac{1}{2}\vartheta$ and $\tfrac{1}{2}(\varepsilon - \varepsilon_1)$ are so small that their cosines may always be put equal to unity, and, consequently, also those of $\tfrac{1}{2}\pi$ and $\tfrac{1}{2}(\psi - \psi_1)$; while for their sines we may substitute the arcs. We thus obtain at once, from the first two equations,

$$\psi - \psi_1 = \vartheta \cos \tfrac{1}{2}(\varepsilon + \varepsilon_1)$$

where we can take, with sufficient accuracy,

$$\begin{aligned} \cos \tfrac{1}{2}(\varepsilon + \varepsilon_1) &= \cos (\varepsilon_0 - 0''.2369\,t) \\ &= \cos \varepsilon_0 + 0''.2369\,t \sin 1'' \sin \varepsilon_0 \end{aligned}$$

and hence, by substituting the values of ϑ and ε_0 from (646),

$$\begin{aligned} \psi - \psi_1 &= 0''.1387\,t - 0''.0002218\,t^2 \\ \psi_1 &= 50''.2411\,t + 0''.0001134\,t^2 \end{aligned} \qquad (648)$$

The sum of the squares of the last two equations of (647) gives

$$\pi^2 = \vartheta^2 \sin^2 \tfrac{1}{2}(\varepsilon + \varepsilon_1) + (\varepsilon - \varepsilon_1)^2$$

in which we may take

$$\sin^2 \tfrac{1}{2}(\varepsilon + \varepsilon_1) = \sin^2 \varepsilon_0 - 0''.2369\,t \sin 1'' \sin 2\varepsilon_0$$

and then, substituting the values of ϑ, ε_0, and $\varepsilon - \varepsilon_1$, we obtain

$$\pi^2 = 0''.228111\,t^2 - 0''.0000033234\,t^3$$

and, by extracting the root,

$$\pi = 0''.4776\,t - 0''.0000035\,t^2 \qquad (649)$$

The quotient of the third equation of (647) divided by the fourth gives

$$\tan (\Pi + \tfrac{1}{2}\psi + \tfrac{1}{2}\psi_1) = \frac{\vartheta}{\varepsilon - \varepsilon_1} \sin \tfrac{1}{2}(\varepsilon + \varepsilon_1)$$

in which we have

$$\begin{aligned} \frac{\vartheta}{\varepsilon - \varepsilon_1} &= \frac{0.15119\,t - 0.00024186\,t^2}{-0.4738\,t - 0.0000875\,t^2} \\ &= -0.3191 + 0.00051636\,t \end{aligned}$$

whence, neglecting π^2 as before,

$$\beta - B = - \pi \sin (L - \Pi) \qquad (653)$$

The values of ψ_1, π, and Π being found for the time $1800 + t$, by means of (648), (649), and (650), the formulæ (652) and (653) determine the required precession in the longitude and latitude, and, consequently, also the mean place of the star for the given date.

870. *To find the precession in longitude and latitude between any two given dates.*—Suppose λ and β are given for $1800 + t$, and λ' and β' are required for $1800 + t'$. Denoting by L and B the longitude and latitude for 1800, we shall have, by (652),

$$\lambda - L = \psi_1 + \pi \tan B \cos (L - \Pi)$$
$$\lambda' - L = \psi_1' + \pi' \tan B \cos (L - \Pi')$$

where ψ_1', π', Π' are the quantities given by (648), (649), and (650) when t' is substituted for t. If we subtract the first of these equations from the second, and at the same time introduce the auxiliaries a and A, determined by the conditions

$$a \sin A = (\pi' + \pi) \sin \tfrac{1}{2} (\Pi' - \Pi)$$
$$a \cos A = (\pi' - \pi) \cos \tfrac{1}{2} (\Pi' - \Pi)$$

we find

$$\lambda' - \lambda = \psi_1' - \psi + a \cos \left(L - \frac{\Pi' + \Pi}{2} - A \right) \tan B$$

and in the same manner, from (653),

$$\beta' - \beta = - a \sin \left(L - \frac{\Pi' + \Pi}{2} - A \right)$$

For the values of A and a we have

$$\tan A = \frac{\pi' + \pi}{\pi' - \pi} \tan \tfrac{1}{2} (\Pi' - \Pi) = \frac{t' + t}{t' - t} \tan \tfrac{1}{2} (\Pi' - \Pi)$$

or, by (650),

$$A = \left(\frac{t' + t}{2} \right) \frac{\Pi' - \Pi}{t' - t} = - 8''.505 \left(\frac{t' + t}{2} \right)$$

so that $\cos A$ may be put equal to unity, and therefore we have

$$a = \pi' - \pi$$

We may also put $\tan \beta$ instead of $\tan B$ in the above formulæ, since the error in $\lambda' - \lambda$ thus produced will be only a term in π^2; and for L we may take $\lambda - \psi_1$: so that if we put

$$L - \frac{\pi' + \pi}{2} - A = \lambda - M$$

and then substitute the numerical values of our constants, we shall have the following formulæ for computing the precession from $1800 + t$ to $1800 + t'$:

$$
\left.
\begin{aligned}
M = \;& 172° \, 45' \, 31'' + t \,.\, 50''.241 - (t' + t) \, 8''.505 \\[4pt]
\lambda' - \lambda = \;& (t' - t) \, [50''.2411 + (t' + t) \, 0''.0001134] \\[4pt]
& + (t' - t) \, [0''.4776 - (t' + t) \, 0''.0000035] \cos (\lambda - M) \tan \beta \\[4pt]
\beta' - \beta = \;& - (t' - t) \, [0''.4776 - (t' + t) \, 0''.0000035] \sin (\lambda - M)
\end{aligned}
\right\} \quad (654)
$$

These are the same as BESSEL's formulæ in the *Tabulæ Regiomontanæ*, except that we have here employed the constants given by PETERS, and the epoch to which t and t' are referred is 1800.

To find the annual precession in longitude for a given date.—If we divide the equations (654) by $t' - t$, the quotients

$$\frac{\lambda' - \lambda}{t' - t}, \qquad \frac{\beta' - \beta}{t' - t}$$

will express the mean annual precession between the two dates; and if we then suppose t' and t to differ by an infinitesimal quantity, or put $t' = t$, these quotients will become the differential coefficients which express the annual precession for the instant $1800 + t$; namely,

$$
\left.
\begin{aligned}
\frac{d\lambda}{dt} = \;& 50''.2411 + 0''.0002268 \, t \\[4pt]
& + [0''.4776 - 0.0000070 \, t] \cos (\lambda - M) \tan \beta \\[4pt]
\frac{d\beta}{dt} = \;& - [0''.4776 - 0.0000070 \, t] \sin (\lambda - M)
\end{aligned}
\right\} \quad (655)
$$

in which

$$M = 172° \, 45' \, 31'' + 33''.23 \, t$$

EXAMPLE.—For the star *Spica*, we have, for the beginning of the year 1800,

the mean longitude, $L = \quad 201° \, 3' \; 5''.97$
the mean latitude, $\quad B = - \quad 2° \, 2' \, 22''.64$

Find its mean longitude and latitude for the beginning of the year 1860.

First. By the direct formulæ (652) and (653).—We find, by (648), (649), and (650), for $t = 60$,

$$\psi_1 = 50' \; 14''.874$$
$$\pi = 28''.6434$$
$$\Pi = 172° \; 37' \; 1''$$

whence

$$L - \Pi = 28° \; 26' \; 5''$$
$$\pi \tan B \cos (L - \Pi) = - \quad 0''.897$$
$$\pi \sin (L - \Pi) = + \; 13''.639$$

and hence, by (652) and (653), the precession is

$$\lambda - L = \quad 50' \; 14''.874 - 0''.897 = 50' \; 13''.977$$
$$\beta - B = - \; 13''.639$$

and the mean longitude and latitude for 1860.0 are

$$\lambda = \quad 201° \; 53' \; 20''.95$$
$$\beta = - \quad 2° \; 2' \; 36''.28$$

Second. By the use of the annual precession.—The mean annual precession for the sixty years from 1800 to 1860 is the annual precession for 1830. Hence, by taking $t = 30$ in (655), and denoting by λ_0 and β_0 the longitude and latitude for 1830,

$$\frac{d\lambda}{dt} = \quad 50''.2479 + 0''.4774 \cos (\lambda_0 - M) \tan \beta_0$$

$$\frac{d\beta}{dt} = - \; 0''.4774 \sin (\lambda_0 - M)$$

$$M = \quad 173° \; 2' \; 8''.$$

To compute these, we can employ approximate values of λ_0 and β_0, found by adding the general precession for thirty years to L, and neglecting the terms in π; namely,

$$\lambda_0 = 201° \; 28'.2 \qquad\qquad \beta_0 = - \; 2° \; 2'.6$$

and hence $\lambda_0 - M = 28° \; 26'.1$,

$$\frac{d\lambda}{dt} = 50''.2329 \qquad\qquad \frac{d\beta}{dt} = - \; 0''.2274$$

These multiplied by 60 give the whole precession from 1800 to 1860,

$$\lambda - L = 50' \; 13''.97 \qquad \beta - B = - \; 13''.64$$

agreeing with the values found above.

371. *Given the mean right ascension and declination of a star for any date* $1800 + t$, *to find the mean right ascension and declination for any other date* $1800 + t'$.—Let $V_1 V_1'$ (Fig. 55) be the fixed ecliptic of 1800, $V_1 Q$ the mean equator of $1800 + t$, $V_1' Q$ the mean equator of $1800 + t'$, Q the intersection of these circles (or the ascending node of the second upon the first). The position of the point Q is found as follows. The arc $V_1 V_1'$ is the luni-solar precession for the interval $t' - t$: so that, distinguishing by accents the quantities obtained by (646) when t' is put for t, we have, in the triangle $Q V_1 V_1'$,
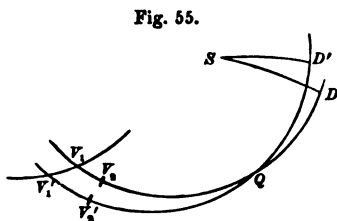
$$V_1 V_1' = \psi' - \psi, \qquad Q V_1 V_1' = 180° - \epsilon_1, \qquad Q V_1' V_1 = \epsilon_1',$$

and putting

$$Q V_1 = 90° - z, \qquad Q V_1' = 90° + z', \qquad V_1 Q V_1' = \Theta,$$

we find, by GAUSS's equations of Sph. Trig.,

$$\left. \begin{array}{l} \cos \tfrac{1}{2}\Theta \sin \tfrac{1}{2}(z' + z) = \sin \tfrac{1}{2}(\psi' - \psi) \cos \tfrac{1}{2}(\epsilon_1' + \epsilon_1) \\ \cos \tfrac{1}{2}\Theta \cos \tfrac{1}{2}(z' + z) = \cos \tfrac{1}{2}(\psi' - \psi) \cos \tfrac{1}{2}(\epsilon_1' - \epsilon_1) \\ \sin \tfrac{1}{2}\Theta \sin \tfrac{1}{2}(z' - z) = \cos \tfrac{1}{2}(\psi' - \psi) \sin \tfrac{1}{2}(\epsilon_1' - \epsilon_1) \\ \sin \tfrac{1}{2}\Theta \cos \tfrac{1}{2}(z' - z) = \sin \tfrac{1}{2}(\psi' - \psi) \sin \tfrac{1}{2}(\epsilon_1' + \epsilon_1) \end{array} \right\} \quad (656)$$

which determine Θ, z, and z' in a rigorous manner. But, since $\tfrac{1}{2}(\epsilon_1' - \epsilon_1)$ is exceedingly small, we can always put unity for its cosine, and the arc for the sine, and, consequently, the same may be done in the case of the arc $\tfrac{1}{2}(z' - z)$; we thus obtain the following simple but accurate formulæ:

$$\left. \begin{array}{l} \tan \tfrac{1}{2}(z' + z) = \tan \tfrac{1}{2}(\psi' - \psi) \cos \tfrac{1}{2}(\epsilon_1' + \epsilon_1) \\[2mm] \tfrac{1}{2}(z' - z) = \dfrac{\tfrac{1}{2}(\epsilon_1' - \epsilon_1)}{\tan \tfrac{1}{2}(\psi' - \psi) \sin \tfrac{1}{2}(\epsilon_1' + \epsilon_1)} \\[2mm] \sin \tfrac{1}{2}\Theta = \sin \tfrac{1}{2}(\psi' - \psi) \sin \tfrac{1}{2}(\epsilon_1' + \epsilon_1) \end{array} \right\} \quad (657)$$

If V_2 and V_2' are the positions of the mean equinox in $1800 + t$ and $1800 + t'$, $V_1 V_2$ is the planetary precession for the first and $V_1' V_2'$ that for the second of these times, which being denoted by ϑ and ϑ' we have

$$V_2 Q = 90° - z - \vartheta$$
$$V_2' Q = 90° + z' - \vartheta'$$
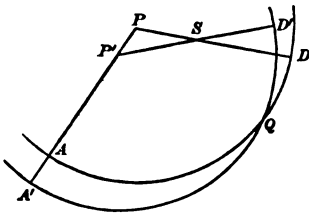
If then we put

$a, \delta =$ the given mean right ascension and declination of a
     star S, for $1800 + t$,

$a', \delta' =$ those required for $1800 + t'$,

we have $a = V_2 D$, and $a' = V_2' D'$ and, consequently,

$$QD = V_2 D - V_2 Q = a + z + \vartheta - 90°,$$
$$QD' = V_2' D' - V_2' Q = a' - z' + \delta' - 90°,$$

Fig. 56.

Now, let P and P' (Fig. 56) be the poles of the equator at the times $1800 + t$, $1800 + t'$, AQD, $A'QD'$, the two positions of the equator at these times, as in Fig. 55; S the star. Q is the pole of the great circle $PP'A'$ joining the poles P and P', and, therefore, $PP' = AA' = AQA' = \Theta$, and in the triangle $PP'S$ we have

$$PS = 90° - \delta, \qquad P'S = 90° - \delta', \qquad PP' = \Theta$$
$$SPP' = \qquad AD = 90° + QD = \qquad a + z + \vartheta$$
$$SP'P = 180° - A'D' = 90° - QD' = 180° - (a' - z' + \vartheta')$$

Hence, by the fundamental equations of Spherical Trigonometry,

$$\left.\begin{array}{l} \cos \delta' \sin (a' - z' + \vartheta') = \cos \delta \sin (a + z + \vartheta) \\ \cos \delta' \cos (a' - z' + \vartheta') = \cos \delta \cos (a + z + \vartheta) \cos \Theta - \sin \delta \sin \Theta \\ \qquad \sin \delta' = \cos \delta \cos (a + z + \vartheta) \sin \Theta + \sin \delta \cos \Theta \end{array}\right\} \quad (658)$$

We have thus a rigorous and direct solution of our problem by finding, first, Θ, z, and z' from (656), and hence a' and δ' by (658), employing the values of ε, ψ, ϑ for the time $1800 + t$, and of ε', ψ', ϑ' for the time $1800 + t'$, as given by (646) for the two dates.

372. The formulæ (658) may be adapted for logarithmic computation by the introduction of an auxiliary angle in the usual manner; or we may employ the GAUSSIAN equations, which, if we denote the angle at the star by C, and for the sake of brevity put

$$A = a + z + \vartheta \qquad\qquad A' = a' - z' + \vartheta' \qquad (659)$$

give

$$\cos \tfrac{1}{2}(90° + \delta') \sin \tfrac{1}{2}(A'+C) = \cos \tfrac{1}{2}(90° + \delta - \Theta) \sin \tfrac{1}{2}A$$
$$\cos \tfrac{1}{2}(90° + \delta') \cos \tfrac{1}{2}(A'+C) = \cos \tfrac{1}{2}(90° + \delta + \Theta) \cos \tfrac{1}{2}A$$
$$\sin \tfrac{1}{2}(90° + \delta') \sin \tfrac{1}{2}(A'-C) = \sin \tfrac{1}{2}(90° + \delta - \Theta) \sin \tfrac{1}{2}A$$
$$\sin \tfrac{1}{2}(90° + \delta') \cos \tfrac{1}{2}(A'-C) = \sin \tfrac{1}{2}(90° + \delta + \Theta) \cos \tfrac{1}{2}A$$

373. We may, however, obtain greater precision by computing the differences between A and A' and between δ and δ'. From the first two equations of (658) we deduce

$$\cos \delta' \sin (A'-A) = \cos \delta \sin A \sin \Theta [\tan \delta + \tan \tfrac{1}{2}\Theta \cos A]$$
$$\cos \delta' \cos (A'-A) = \cos \delta - \cos \delta \cos A \sin \Theta [\tan \delta + \tan \tfrac{1}{2}\Theta \cos A]$$

so that, if we put

$$p = \sin \Theta (\tan \delta + \tan \tfrac{1}{2}\Theta \cos A)$$

we have

$$\tan (A' - A) = \frac{p \sin A}{1 - p \cos A}$$

and, by NAPIER's Analogy,*

$$\tan \tfrac{1}{2}(\delta' - \delta) = \tan \tfrac{1}{2}\Theta \, \frac{\cos \tfrac{1}{2}(A' + A)}{\cos \tfrac{1}{2}(A' - A)}$$

(660)

EXAMPLE.—The mean place of *Polaris* for 1755, according to the *Tabulæ Regiomontanæ*, is

$$\alpha = 10° 55' 44''.955 \qquad \delta = 87° 59' 41''.12$$

it is required to reduce this place to the mean equator and equinox of 1820.

For 1755 we take $t = -45$; and for 1820, $t' = +20$; and, by (646), we find—

For 1755.	For 1820.
$\psi = -37' 47''.31$	$\psi' = +16' 47''.55$
$\vartheta = \quad -7''.29$	$\vartheta' = \quad +2''.93$
$\epsilon_1 = 23° 27' 54''.23488$	$\epsilon_1' = 23° 27' 54''.22294$

and hence

$$\tfrac{1}{2}(\psi' - \psi) = 27' 17''.43$$
$$\tfrac{1}{2}(\epsilon_1' - \epsilon_1) = - 0''.00597$$
$$\tfrac{1}{2}(\epsilon' + \epsilon) = 23° 27' 54''.23$$

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* The formulæ (657), (658), (659), (660) are those given by BESSEL in the *Tabulæ Regiomontanæ*.

with which the formulæ (657) give

$$\tfrac{1}{2}(z' + z) = + 25'\ 2''.02$$
$$\tfrac{1}{2}(z' - z) = \qquad - 1''.89$$
$$z = + 25'\ 3''.91$$
$$z' = + 25'\ 0''.13$$
$$\log \sin \tfrac{1}{2}\Theta = 7.499823$$

Then, by the formulæ (660), we find

$$A = a + z + \vartheta = 11°\ 20'\ 41''.57$$

$\log p$	9.256676	$\log \tan \tfrac{1}{2}\Theta$ 7.499825
$\log \sin A$	9.293836	$\log \cos \tfrac{1}{2}(A' + A)$ 9.989446
$\log \cos A$	9.991430	$\log \sec \tfrac{1}{2}(A' - A)$ 0.000101
$\log p \cos A$	9.248106	$\log \tan \tfrac{1}{2}(\delta' - \delta)$ 9.489372

ar. co. $\log(1 - p\cos A)$ 0.084629

$\log \tan(A' - A)$ 8.635141

$$A' - A = 2°\ 28'\ 18''.08 \qquad \delta' - \delta = \qquad 21'\ 12''.99$$
$$A' = 13°\ 48'\ 59''.65$$
$$a' = A' + z' - \vartheta' = 14°\ 13'\ 56''.85 \qquad \delta' = 88°\ 20'\ 54''.11$$

374. *To find the annual precession in right ascension and declination.*—In computing the precession for a single year, the square of Θ becomes insensible, and we may take, instead of (660), the approximate formula

$$A' - A = a' - a - (z' + z) + \vartheta' - \vartheta = \Theta \sin a \tan \delta$$

and from (657) we then have, with sufficient accuracy,

$$z' + z = (\psi' - \psi)\cos \epsilon_1$$
$$\Theta = (\psi' - \psi)\sin \epsilon_1$$

Substituting these values in the above formula, and then dividing by $t' - t$, we have

$$\frac{a' - a}{t' - t} = \frac{\psi' - \psi}{t' - t}\cos \epsilon_1 - \frac{\vartheta' - \vartheta}{t' - t} + \frac{\psi' - \psi}{t' - t}\sin \epsilon_1 \sin a \tan \delta$$

which gives the *annual* precession between the times $1800 + t$ and $1800 + t'$, the unit of time being one year. But, in order that the formula may express the rate of change at the instant $1800 + t$, we must suppose the interval $t' - t$ to become infinitely small; that is, we must write the formula thus:

$$\frac{d\alpha}{dt} = \frac{d\psi}{dt} \cos \epsilon_1 - \frac{d\vartheta}{dt} + \frac{d\psi}{dt} \sin \epsilon_1 \sin \alpha \tan \delta$$

and similarly, from the last equation of (660),

$$\frac{d\delta}{dt} = \frac{d\psi}{dt} \sin \epsilon_1 \cos \alpha$$

Putting then

$$m = \frac{d\psi}{dt} \cos \epsilon_1 - \frac{d\vartheta}{dt}$$

$$n = \frac{d\psi}{dt} \sin \epsilon_1$$

$$\left.\right\} \quad (661)$$

we find, by (646),

$$\frac{d\psi}{dt} \cos \epsilon_1 = (50''.3798 - 0''.0002168\, t) \cos \epsilon_1$$

$$= 46''.2135 - 0''.00019887\, t$$

$$\frac{d\vartheta}{dt} = \quad 0''.1512 - 0''.00048372\, t$$

and hence

$$m = 46''.0623 + 0''.0002849\, t$$
$$n = 20''.0607 - 0''.0000863\, t$$

$$\left.\right\} \quad (662)$$

and the annual precession in right ascension and declination for the time 1800 + t is found by the formulæ

$$\frac{d\alpha}{dt} = m + n \sin \alpha \tan \delta$$

$$\frac{d\delta}{dt} = n \cos \alpha$$

$$\left.\right\} \quad (663)$$

These formulæ may be used for computing the whole precession between any two dates, if we multiply the annual precession at the *middle time* between the two dates by the number of years in the interval.

EXAMPLE.—The mean right ascension and declination of *Spica* for 1800 are, by the *Tabulæ Regiomontanæ*,

$$\alpha = \quad 13^h\, 14^m\, 40^s.5057$$
$$\delta = -10°\, 6'\, 46''.843$$

Find the mean right ascension (α') and declination (δ') for 1860. We have, for 1830, by making $t = 30$ in (662),

$$m = 46''.0708 \qquad\qquad n = 20''.0581$$

and, for a first approximation, taking $\alpha' = \alpha$, $\delta' = \delta$, we have, by (663),

$$\frac{d\alpha}{dt} = + 47''.22 \qquad\qquad \frac{d\delta}{dt} = - 19''.00$$

The approximate precession for sixty years is, therefore,

in R. A., $+ 2833'' = + 188^{s}.9$ in dec., $- 1140''$

which, applied to α and δ, give the approximate values for 1860,

$$\alpha' = 13^h\ 17^m\ 49^s.4 \qquad\qquad \delta' = - 10°\ 25'\ 47''$$

The means between these values and those of α and δ are

$$\alpha_0 = 13^h\ 16^m\ 15^s. \qquad\qquad \delta_0 = - 10°\ 16'\ 17''$$

which being employed in (663) give the more correct annual precession for 1830,

$$\frac{d\alpha}{dt} = + 47''.2579 \qquad\qquad \frac{d\delta}{dt} = - 18''.9582$$

The true precession for sixty years is then

in R. A., $+ 2835''.474 = 3^m\ 9^s.0316,$ in dec., $- 18'\ 57''.492,$

which applied to α and δ give the mean place for 1860,

$$\alpha' = 13^h\ 17^m\ 49^s.5373 \qquad\qquad \delta' = - 10°\ 25'\ 44''.335\text{\]}$$

and these values agree almost precisely with those found by the rigorous method of Art. 371.

875. *To find the position of the pole of the equator at a given time.*— The precession causes the pole of the equator to revolve about the pole of the ecliptic (nearly) in a small circle whose polar distance is equal to the obliquity of the ecliptic. The time in which the pole will make a complete revolution and return to the same position (small changes in the obliquity of the ecliptic not considered) is the value of t given by the equation

$$50''.2411t + 0''.0001134t^2 = 360° \times 60 \times 60 = 1296000''$$

which gives

$$t = 24447 \text{ years};$$

or, in round numbers, since the precession is not known with

sufficient precision to determine so great a period exactly, $t = 24500$ years.

To find the position of the pole for any indeterminate time $1800 + t'$, we have only to observe that if P, in Fig. 56, is the pole for a fixed time $1800 + t$, P' that for the time $1800 + t'$, the right ascension of P', reckoned from the equinox of $1800 + t$, is equal to that of the point Q diminished by $90°$. The right ascension of Q is $V_2 Q$ in Fig. 55, and, in Art. 371, we have found

$$V_2 Q = 90° - z - \vartheta$$

Hence, if we put

$A, D =$ the right ascension and declination of the pole at the
   time $1800 + t'$, referred to the equator and equinox
   of $1800 + t$,

we have

$$A = - z - \vartheta$$
$$D = 90° - \Theta$$

which will become known by computing ψ, ψ', ε, ε', ϑ for the times $1800 + t$, $1800 + t'$, and then z and Θ by (657)

An approximate solution is obtained by neglecting the variation of ε, and, consequently, taking $z' = z$, and also neglecting ϑ: so that

$$\left. \begin{array}{l} \tan A = - \tan \tfrac{1}{2} (\psi' - \psi) \cos \varepsilon_0 \\ \sin (45° - \tfrac{1}{2} D) = \sin \tfrac{1}{2} (\psi' - \psi) \sin \varepsilon_0 \end{array} \right\} \quad (664)$$

The ambiguity in determining A by its tangent is removed by observing that $\cos A$ and $\cos \tfrac{1}{2} (\psi' - \psi)$ must have the same sign so long as $\psi' - \psi$ does not exceed $360°$, as we readily infer from the equations (656).

For example, if we wish to find the position of the pole for the year 14000, referred to the equinox of 1850, we take $t = 50$, $t' = 12200$; whence $\psi' - \psi = 165° 33'$, and

$$A = 277° 52' \qquad\qquad D = 43° 28'$$

The position of α *Lyræ* for 1850 is

$$a = 277° 58' \qquad\qquad \delta = 38° 39'$$

consequently, this star, in the year 14000, will be within five degrees of the pole, and will become the pole star of that period.

878. *To reduce a star's mean place from one epoch to another, when the proper motion is given.*—Let α, δ, be the given place for $1800 + t$, and let the given *annual* proper motion in right ascension and declination, referred to the equinox of this date, be denoted by $d\alpha$ and $d\delta$. To reduce to the date $1800 + t'$, we first find the whole proper motion in the interval, by the formulæ

$$\Delta\alpha = d\alpha\,(t' - t) \qquad\qquad \Delta\delta = d\delta\,(t' - t)$$

Then, putting

$$(\alpha) = \alpha + \Delta\alpha \qquad\qquad (\delta) = \delta + \Delta\delta$$

we compute the precession by the formulæ of Arts. 371 to 374, employing in these formulæ (α) and (δ) for α and δ.

If the proper motion $(\Delta\alpha', \Delta\delta')$ had been given for the epoch $1800 + t'$, we should first have computed the precession with the given values α and δ, and, having applied it, if (α') and (δ') were the resulting values, we should have finally $\alpha' = (\alpha') + \Delta\alpha'$, $\delta' = (\delta') + \Delta\delta'$.

879. *To reduce the proper motion in right ascension and declination from one epoch to another.*—If, in Fig. 56, P and P' are the poles of the equator for the epochs $1800 + t$ and $1800 + t'$ respectively, and we suppose the star S to vary its position, the present merely requires us to deduce the relations between the variations of the parts of the triangle SPP', the side PP' being the only constant part. Observing the notation of Art. 371, we have (since ζ, ϑ, ζ', ϑ' do not depend upon the star's place)

$$d(SPP') = d(\alpha + \zeta + \vartheta) = d\alpha$$
$$d(SP'P) = d(180° - \alpha' + \zeta' - \vartheta') = - d\alpha'$$
$$d(SP) = - d\delta$$
$$d(SP') = - d\delta'$$

and hence, by the formulæ (47) and (46), putting γ for the angle at the star,

$$\left.\begin{array}{rl} \cos \delta'.\,d\alpha' = & d\alpha \cos \delta \cos \gamma + d\delta \sin \gamma \\ d\delta' = & - d\alpha \cos \delta \sin \gamma + d\delta \cos \gamma \end{array}\right\} \quad (665)$$

in which

$$\sin \gamma = \frac{\sin \Theta \sin (\alpha + z + \vartheta)}{\cos \delta'} = \frac{\sin \Theta \sin (\alpha' - z' + \vartheta')}{\cos \delta}$$

$$\cos \gamma = \frac{\cos \Theta - \sin \delta \sin \delta'}{\cos \delta \cos \delta'}$$

and hence for 1755 we find

$$d\alpha = + 1''.08228 \qquad\qquad d\delta = + 0''.004098$$

If, now, there had been given both the mean place and the proper motion for 1755, namely,

$$\begin{aligned}
\alpha &= 10°\ 55'\ 44''.955 & \delta &= 87°\ 59'\ 41''.12\\
d\alpha &= + 1''.08228 & d\delta &= + 0''.004098
\end{aligned}$$

to find the mean place for 1820, we should first take

$$\begin{aligned}
(\alpha) &= 10°\ 55'\ 44''.955 + 1''.08228 \times 65 = 10°\ 56'\ 55''.30\\
(\delta) &= 87\quad 59\quad 41\ .12\ + 0\ .004098 \times 65 = 87\quad 59\quad 41\ .39
\end{aligned}$$

and employing these values, instead of α and δ, in (659) and (660), we should find

$$\begin{aligned}
\alpha + z + \vartheta = A &= 11°\ 21'\ 51''.92\\
\log p &= 9.256691\\
A' - A &= 2°\ 28'\ 33''.45\\
\tfrac{1}{2}(\delta' - \delta) &= \quad 10'\ 36''.44
\end{aligned}$$

whence

$$\alpha' = 14°\ 15'\ 22''.57 \qquad\qquad \delta' = 88°\ 20'\ 54''.27$$

as given above.

380. *The proper motion on a great circle.*—If we denote this by ρ, and the angle which the great circle makes with the circle of declination of the star by χ, reckoning the angle from the north towards the east, we have

$$\rho \sin \chi = \Delta\alpha \cos \delta \qquad\qquad \rho \cos \chi = \Delta\delta$$

Thus, we find, in the preceding example, for *Polaris* in 1755,

$$\rho = 0''.03809 \qquad\qquad \chi = 83°\ 49'.4$$

and in 1820,

$$\rho = 0''.03809 \qquad\qquad \chi' = 86°\ 17'.8$$

where the accent of χ' is used for the second epoch, but ρ is necessarily the same for both epochs.

It is evident, moreover, that we have $\chi' = \chi + \gamma$, and hence, if ρ and χ have been found for one epoch, it is only necessary to compute γ to obtain the reduction to another epoch; for we then have, by (665),

$$\begin{aligned}
\cos \delta'\, d\alpha' &= \rho \sin (\chi + \gamma) = \rho \sin \chi'\\
d\delta' &= \rho \cos (\chi + \gamma) = \rho \cos \chi'
\end{aligned}$$

$\varepsilon_0 = 23° 27' 54''.2$. These coefficients, however, are not absolutely constant: so that, according to PETERS, the formulæ for 1900 will be

$$\Delta\varepsilon = \quad 9''.2240\cos\Omega - 0''.0896\cos2\Omega + 0''.0885\cos2\mathbb{C}$$
$$+ 0''.5507\cos2\odot + 0''.0092\cos(\odot + \Gamma)$$

$$(667)$$

$$\Delta\lambda = -17''.2577\sin\Omega + 0''.2073\sin2\Omega - 0''.2041\sin2\mathbb{C} + 0''.0677\sin(\mathbb{C} - \Gamma')$$
$$- 1''.2695\sin2\odot + 0''.1275\sin(\odot - \Gamma) - 0''.0213\sin(\odot + \Gamma)$$

Since the attractions of the sun and moon upon the earth do not disturb the position of the ecliptic, but only that of the equator and its intersection with the ecliptic, the nutation does not affect the latitudes of stars, and its effect upon their longitudes is simply to increase them all by the same quantity $\Delta\lambda$.

382. *To find the nutation in right ascension and declination for a given star at a given time.*—Let α and δ denote the mean right ascension and declination of the star at the given time; α' and δ' the *true* right ascension and declination at this time, or the mean place corrected for the nutation. Let the coefficients of the formulæ for $\Delta\varepsilon$ and $\Delta\lambda$ be found for the given year by interpolation between the values for 1800 and 1900, and then, taking $\Omega, \mathbb{C}, \odot, \Gamma$, and Γ' from the Ephemeris for the given date (the day of the year, and, for the greatest precision, the hour of the day), we can compute the values of $\Delta\varepsilon$ and $\Delta\lambda$. We can then have either a rigorous or an approximate solution of our problem.

A rigorous solution may be obtained by employing the formulæ (656), (658), and (659), substituting $\varepsilon + \frac{1}{2}\Delta\varepsilon$, $\Delta\varepsilon$, $\Delta\lambda$, $\alpha + z$, and $\alpha' - z'$ for $\frac{1}{2}(\varepsilon_1' + \varepsilon_1)$, $\varepsilon_1' - \varepsilon_1$, $\psi' - \psi$, A and A', respectively.

Another rigorous solution is obtained by first computing the mean longitude λ and latitude β, from the given α and δ, and the mean obliquity ε, by Art. 23. Then, with the true longitude $\lambda + \Delta\lambda$, the true latitude β, and the true obliquity $\varepsilon + \Delta\varepsilon$, we can compute the true right ascension α' and declination δ' by Art. 26.

But, since $\Delta\varepsilon$ and $\Delta\lambda$ are very small, an approximate solution by means of differential formulæ will be sufficiently accurate, and practically more convenient. The effect of varying λ and ε by $\Delta\lambda$ and $\Delta\varepsilon$, while β is constant, is, by the equations (53),

$$\alpha' - \alpha = \Delta\lambda \cdot \frac{\cos\eta\cos\beta}{\cos\delta} - \Delta\varepsilon\cos\alpha\tan\delta$$
$$\delta' - \delta = \Delta\lambda \cdot \sin\eta\cos\beta + \Delta\varepsilon\sin\alpha$$

in which η is the position angle at the star, as defined in Art. 25.

Substituting the values of $\cos \eta \cos \beta$ and $\sin \eta \cos \beta$ there given, we have

$$a' - a = \Delta\lambda \, (\cos \epsilon + \sin \epsilon \sin a \tan \delta) - \Delta\epsilon \cos a \tan \delta$$
$$\delta' - \delta = \Delta\lambda \sin \epsilon \cos a + \Delta\epsilon \sin a$$

Hence, substituting the values of $\Delta\lambda$ and $\Delta\epsilon$ for 1800, with $\epsilon = 23° 27' 54''$, and also the values for 1900 with $\epsilon = 23° 27' 7''$, we find

$a' - a =$
$- (15''.8148 + 6''.8650 \sin a \tan \delta) \sin \Omega \quad - 9''.2231 \cos a \tan \delta \cos \Omega$
  $\quad\quad 15''.8321 \quad\quad 6''.8682 \quad\quad\quad\quad\quad\quad\quad\quad 9''.2240$
$+ (\;\; 0''.1902 + 0''.0825 \sin a \tan \delta) \sin 2\Omega + 0''.0897 \cos a \tan \delta \cos 2\Omega$
$- (\;\; 0''.1872 + 0''.0813 \sin a \tan \delta) \sin 2\mathbb{C} - 0''.0886 \cos a \tan \delta \cos 2\mathbb{C}$
$+ (\;\; 0''.0621 + 0''.0270 \sin a \tan \delta) \sin (\mathbb{C} - \Gamma')$
$- (\;\; 1''.1644 + 0''.5055 \sin a \tan \delta) \sin 2\odot - 0''.5510 \cos a \tan \delta \cos 2\odot$
$+ (\;\; 0''.1173 + 0''.0509 \sin a \tan \delta) \sin (\odot - \Gamma)$
$- (\;\; 0''.0195 + 0''.0085 \sin a \tan \delta) \sin (\odot + \Gamma) - 0''.0098 \cos a \tan \delta \cos (\odot + \Gamma)$

(668)

$\delta' - \delta = - \;\; 6''.8650 \cos a \sin \Omega \quad + 9''.2231 \sin a \cos \Omega$
  $\quad\quad\quad\quad 6''.8682 \quad\quad\quad\quad\quad\quad 9''.2240$
$+ 0''.0825 \cos a \sin 2\Omega - 0''.0897 \sin a \cos 2\Omega$
$- 0''.0813 \cos a \sin 2\mathbb{C} + 0''.0886 \sin a \cos 2\mathbb{C}$
$+ 0''.0270 \cos a \sin (\mathbb{C} - \Gamma')$
$- 0''.5055 \cos a \sin 2\odot + 0''.5510 \sin a \cos 2\odot$
$+ 0''.0509 \cos a \sin (\odot - \Gamma)$
$- 0''.0085 \cos a \sin (\odot + \Gamma) + 0''.0098 \sin a \cos (\odot + \Gamma)$

The values of the coefficients which sensibly change during the century are given for 1900 in small figures below the values for 1800.*

Previous to the investigations of PETERS, the only terms retained in the nutation formula were those depending on $\Omega, 2\Omega, 2\mathbb{C}$, and $2\odot$. Of the additional terms added by him, I have retained only those which can have any sensible effect in the actual state of the art of astronomical observation.

383. *General tables for the nutation in right ascension and declination.*—Of the various tables proposed for facilitating the compu-

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* If we take into account the squares of $\Delta\lambda$ and $\Delta\epsilon$ and their product in the development of $a' - a$ and $\delta' - \delta$ in series, some of the coefficients are changed, but only by two or three units in the last decimal place. Compare the formulæ of the text with those given by PETERS in the *Numerus Constans*, and by STRUVE in the *Astronom. Nach.*, No. 486.

tation of the nutation formulæ, the most compendious are those computed by NICOLAI, according to the form suggested by GAUSS, and included in WARNSTORFF's edition of SCHUMACHER's *Hülfs-tafeln*. In these tables the new constants are adopted from PETERS, as in the preceding formulæ, and the epoch is 1850.

For the *lunar* nutation in right ascension, the first table gives, with the argument Ω, the quantity

$$- 15''.8235 \sin \Omega = c$$

The two remaining terms in the first line of our formula are reduced to a single term by assuming auxiliaries b and B, also given in the tables with the argument Ω, determined by the conditions

$$b \sin (\Omega + B) = 6''.8666 \sin \Omega$$
$$b \cos (\Omega + B) = 9''.2235 \cos \Omega$$

Thus, the first line of the formula, containing the principal terms of the lunar nutation in right ascension, becomes

$$c - b \cos (\Omega + B - a) \tan \delta$$

By the use of the same auxiliaries, the first two terms of the lunar nutation in declination are reduced to the following:

$$- b \sin (\Omega + B - a)$$

For the solar nutation, the second table gives, with the argument $2\odot$, the quantity

$$- 1''.1644 \sin 2\odot = g$$

and the two remaining terms involving $2\odot$ are reduced to a single one by the auxiliaries f and F, given in the table, which are determined by the conditions

$$f \sin (2\odot + F) = 0''.5055 \sin 2\odot$$
$$f \cos (2\odot + F) = 0''.5510 \cos 2\odot$$

so that the solar nutation in right ascension is

$$g - f \cos (2\odot + F - a) \tan \delta$$

and the solar nutation in declination is

$$- f \sin (2\odot + F - a)$$

Almost all the remaining terms of the formulæ may also be found by means of the table for the solar nutation. The coefficients of the terms in 2Ω and $2\mathbb{C}$ are about one-sixth part of

so small an interval to be rectilinear and uniform, and the motion of light to be uniform, the lines BA and $B'A'$ are parallel, and the ray of light during its progress from A to B', is constantly in the axis of the telescope; for instance, when the telescope is in the position ba, the ray will have reached the point a, and we have

$$Aa : Bb = AB' : BB'$$

The difference of apparent direction thus caused by the motion of the earth combined with that of light is called the *aberration of the fixed stars*. When we also take into account the motion of the luminous body, as in the case of the planets, another species of aberration occurs, which will be considered hereafter, under the name of the *planetary* aberration.

The whole displacement of the star produced by aberration is in the plane passed through the star and the line of the observer's motion, and the star appears to be thrown *forward* in this plane in the direction of that motion. Thus, in the figure the whole aberration is the angle $SB'A'$; and, if we conceive the plane of the lines SB' and BB' to be produced to the celestial sphere, this plane will be that of a great circle drawn through the place of the star and the points of the sphere in which the line BB' meets it. The displacement of the star will be the arc of this circle subtending the angle $SB'A'$ and measured from the star towards that point of the sphere towards which the observer is moving.

385. *To find the aberration of a star in the direction of the observer's motion.*—Let

$\vartheta = AB'B_{1} =$ the true direction of the star referred to the line $B'B_{1}$,

$\quad =$ the arc of a great circle of the sphere joining the star's true place and the point *from* which the observer is moving,

$\vartheta' =$ the apparent direction of the star referred to the same line, $= ABB_{1}$,

$V =$ the velocity of light,

$v =$ the velocity of the observer;

then the aberration in the plane of motion is the angle $A'B'A = B'AB = \vartheta' - \vartheta$, and the triangle ABB' gives

$$\frac{\sin (\vartheta' - \vartheta)}{\sin \vartheta'} = \frac{BB'}{AB'} = \frac{v}{V}$$

As $\vartheta' - \vartheta$ is very small, we may put the arc for the sine; and if we then also put

$$k = \frac{v}{V \sin 1''} \qquad (669)$$

we shall have

$$\vartheta' - \vartheta = k \sin \vartheta' \qquad (670)$$

where the constant k may be regarded as known from the velocities of light and of the observer.

386. The motion of the observer on the surface of the earth is the resultant of the motion of the earth in its orbit and its rotation on its axis; that is, of its *annual* and *diurnal* motions. These may be separately considered.

The *annual aberration* is that part of the total aberration which results from the earth's annual motion. It may be called the aberration for the earth's centre.

The *diurnal aberration* is that part of the total aberration which results from the earth's diurnal motion. It obviously varies with the position of the observer on the earth's surface, and vanishes for an observer at the poles.

387. *To find the annual aberration of a star in longitude and latitude.*—Let

λ, β = the true longitude and latitude of the star,
λ', β' = the apparent longitude and latitude (affected by aberration),
$\odot$ = the true longitude of the sun.

The point of the sphere from which the earth appears to be moving is a point in the ecliptic whose longitude is $90° + \odot$ (the eccentricity of the earth's orbit being here neglected), and the mean velocity of the earth in its orbit may be supposed to be substituted in (669): so that k is known.

Fig. 58.

If, then, BE (Fig. 58) is an arc of the ecliptic, E the point from which the earth is moving, S the true place of the star, and if SB is drawn perpendicular to BE, we have, in the right triangle SBE,

$$SB = \beta, \qquad BE = 90° + \odot - \lambda, \qquad SE = \vartheta,$$
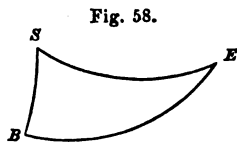
and hence, if we denote the angle E by γ, we have

$$\left.\begin{array}{rl} \sin \vartheta \sin \gamma = & \sin \beta \\ \sin \vartheta \cos \gamma = & \cos \beta \cos (\odot - \lambda) \\ \cos \vartheta = & -\cos \beta \sin (\odot - \lambda) \end{array}\right\} \quad (671)$$

The apparent place of the star is on the great circle ES at the distance ϑ' from S: so that, if we now suppose S to be the apparent place, the angle γ is not changed, and we have

$$\left.\begin{array}{rl} \sin \vartheta' \sin \gamma = & \sin \beta' \\ \sin \vartheta' \cos \gamma = & \cos \beta' \cos (\odot - \lambda') \\ \cos \vartheta' = & -\cos \beta' \sin (\odot - \lambda') \end{array}\right\} \quad (672)$$

If, then, the true place of the star is given, the equations (671) may be used to determine γ and ϑ; then ϑ' will be found from (670), and, finally, λ' and β' will be found from (672). This is the direct and rigorous solution of the problem; but a more convenient solution is obtained by eliminating ϑ and γ as follows. We find, from the equations (671) and (672),

$$\sin \vartheta' \cos \vartheta \cos \gamma = -\cos \beta \cos \beta' \sin (\odot - \lambda) \cos (\odot - \lambda')$$
$$\sin \vartheta \cos \vartheta' \cos \gamma = -\cos \beta \cos \beta' \cos (\odot - \lambda) \sin (\odot - \lambda')$$

the difference of which is

$$\sin (\vartheta' - \vartheta) \cos \gamma = -\cos \beta \cos \beta' \sin (\lambda' - \lambda)$$

whence

$$\lambda' - \lambda = -\frac{(\vartheta' - \vartheta) \cos \gamma}{\cos \beta \cos \beta'} = -\frac{k \sin \vartheta' \cos \gamma}{\cos \beta \cos \beta'}$$

or

$$\lambda' - \lambda = -k \frac{\cos (\odot - \lambda')}{\cos \beta} \quad (673)$$

Again, we find, from our equations,

$$\cot \gamma = \cot \beta' \cos (\odot - \lambda') = \cot \beta \cos (\odot - \lambda)$$

by which β' can be found from β after λ' has been found by (673), or we may find the difference between β' and β thus:

$$\tan \beta' - \tan \beta = \tan \beta' \left[\frac{\cos (\odot - \lambda') - \cos (\odot - \lambda)}{\cos (\odot - \lambda')} \right]$$

$$\sin (\beta' - \beta) = \frac{2 \sin \tfrac{1}{2} (\lambda' - \lambda) \sin [\odot - \tfrac{1}{2} (\lambda' + \lambda)] \sin \beta' \cos \beta}{\cos (\odot - \lambda')}$$

whence, taking $2 \sin \tfrac{1}{2} (\lambda' - \lambda) = \sin (\lambda' - \lambda)$, we obtain, by means of (673),

$$\beta' - \beta = -k \sin [\odot - \tfrac{1}{2} (\lambda' + \lambda)] \sin \beta' \quad (674)$$

sphere being unity. The period in which a star appears to describe this ellipse is a sidereal year.

389. *To find the annual aberration in right ascension and declination.*—Let

$A, D =$ the right ascension and declination of the point E
  (from which the earth is moving);

then, in the triangle formed by the point E, the star, and the pole of the equator, the sides are $90° - D$, $90° - \delta$, and ϑ; and the angle opposite to ϑ is $A - \alpha$. If then we suppose the side ϑ to vary, the corresponding variations of the angle $A - \alpha$ and the side $90° - \delta$ may be directly deduced by the differential formulæ of Art. 34. The angle at E and the side $90° - D$ being constant, we find

$$\cos \delta \cdot d\alpha = - d\vartheta \sin C$$
$$d\delta = - d\vartheta \cos C$$

where C denotes the angle at the star. For determining C, our triangle gives

$$\sin \vartheta \sin C = \cos D \sin (A - \alpha)$$
$$\sin \vartheta \cos C = \cos \delta \sin D - \sin \delta \cos D \cos (A - \alpha)$$

In (670) we may employ $\sin \vartheta$ for $\sin \vartheta'$: so that, putting $\alpha' - \alpha$ and $\vartheta' - \vartheta$ for $d\alpha$ and $d\vartheta$, we find

$$\left. \begin{aligned} \alpha' - \alpha &= - k \sec \delta \cos D \sin (A - \alpha) \\ \delta' - \delta &= - k \left[\cos \delta \sin D - \sin \delta \cos D \cos (A - \alpha) \right] \end{aligned} \right\} \quad (676)$$

The quantities A and D are found from the right triangle formed by the equator, the ecliptic, and the declination circle drawn through E, by the formulæ,

$$\left. \begin{aligned} \cos D \cos A &= - \sin \odot \\ \cos D \sin A &= \cos \odot \cos \epsilon \\ \sin D &= \cos \odot \sin \epsilon \end{aligned} \right\} \quad (677)$$

If we substitute these values in the formulæ for $\alpha' - \alpha$ and $\delta' - \delta$, after developing $\sin (A - \alpha)$ and $\cos (A - \alpha)$, we obtain

$$\left. \begin{aligned} \alpha' - \alpha &= - k \sec \delta (\cos \odot \cos \epsilon \cos \alpha + \sin \odot \sin \alpha) \\ \delta' - \delta &= - k \cos \odot (\sin \epsilon \cos \delta - \cos \epsilon \sin \delta \sin \alpha) \\ & \quad - k \sin \odot \sin \delta \cos \alpha \end{aligned} \right\} \quad (678)$$

If we retain the terms of the second order, (omitting, however, those which do not involve ⊙, or the non-periodic terms), we find that the aberration in right ascension obtains the additional terms

$$- \tfrac{1}{4} k^2 \sin 1'' (1 + \cos^2 \epsilon) \cos 2\odot \sin 2\alpha \sec^2 \delta$$
$$+ \tfrac{1}{2} k^2 \sin 1'' \cos \epsilon \sin 2\odot \cos 2\alpha \sec^2 \delta$$

and the aberration in declination the terms

$$+ \tfrac{1}{8} k^2 \sin 1'' [\sin^2 \epsilon - (1 + \cos^2 \epsilon) \cos 2\odot \cos 2\alpha] \tan \delta$$
$$- \tfrac{1}{4} k^2 \sin 1'' \cos \epsilon \sin 2\odot \sin 2\alpha \tan \delta$$

Substituting the value of k in these terms, together with $\epsilon = 23° 27' 30''$ (for 1850), and omitting insensible quantities, the corrections of the formulæ (678) will be

$$\left. \begin{array}{ll} \text{in } (\alpha' - \alpha), & - 0''.000931 \sin 2(\odot - \alpha) \sec^2 \delta \\ \text{in } (\delta' - \delta), & - 0''.000466 \cos 2(\odot - \alpha) \tan \delta \end{array} \right\} \text{(678*)}$$

EXAMPLE 1.—The mean longitude and latitude of *Spica* for January 10, 1860, are

$$\lambda = 201° 53' 22''.33 \qquad\qquad \beta = - 2° 2' 36''.29$$

and the sun's longitude is

$$\odot = 289° 30'$$

Hence, we find, by (675), the aberration in longitude and latitude,

$$\lambda' - \lambda = - 0''.85 \qquad\qquad \beta' - \beta = + 0''.73$$

The corresponding mean right ascension and declination are

$$\alpha = 13^h 17^m 49^s.62 \qquad\qquad \delta = - 10° 25' 44''.9$$

whence, by (678), taking $\epsilon = 23° 27'.4$, we find the aberration in right ascension and declination,

$$\alpha' - \alpha = - 0''.53 = - 0^s.035 \qquad\qquad \delta' - \delta = + 0''.99$$

EXAMPLE 2.—The mean place of *Polaris* for 1820.0 was

$$\alpha = 0^h 57^m 1^s.505 = 14° 15' 22''.57$$
$$\delta = 88° 20' 54''.27$$

and for this date,

$$\odot = 280° 0' \qquad\qquad \epsilon = 23° 27'.8$$

with which the aberration in right ascension and declination is found, by (678), to be

$$\alpha' - \alpha = + 62''.51 = + 4^{s}.167 \qquad \delta' - \delta = + 20''.27$$

The additional terms of (678*) are in this case $- 0''.158 =$ $- 0^{s}.011$ and $+ 0''.016$, and the more correct values are, therefore,

$$\alpha' - \alpha = + 4^{s}.156 \qquad \delta' - \delta = + 20''.29$$

390. *Gauss's Tables for computing the aberration in right ascension and declination.*—If we determine a and A by the conditions

$$a \sin (\odot + A) = k \sin \odot$$
$$a \cos (\odot + A) = k \cos \odot \cos \epsilon$$

the formulæ (678) may be expressed as follows:

$$\alpha' - \alpha = - a \sec \delta \cos(\odot + A - \alpha)$$
$$\delta' - \delta = - a \sin \delta \sin (\odot + A - \alpha) - k \cos \odot \cos \delta \sin \epsilon$$
$$= - a \sin \delta \sin (\odot + A - \alpha) - \tfrac{1}{2} k \sin \epsilon \cos (\odot + \delta)$$
$$- \tfrac{1}{2} k \sin \epsilon \cos (\odot - \delta)$$

The first of the tables proposed by GAUSS* gives A and $\log a$ with the argument sun's longitude, and with these quantities we readily compute the aberration in right ascension and the first part of the aberration in declination. The second and third parts of the aberration in declination are taken directly from the second table with the arguments $\odot + \delta$ and $\odot - \delta$. The tables have been recomputed by NICOLAI with the constant $k = 20''.4451$, and are given in WARNSTORFF's edition of SCHUMACHER's *Hülfstafeln.*

The value of ϵ for 1850 is employed in computing these tables. The rate of change of ϵ is so slow that the tables will answer for the whole of the present century, unless more than usual precision is desired.

391. In the preceding investigation of the aberration formulæ we have, for greater simplicity, assumed the earth's orbit to be a circle and its motion in the orbit uniform. Let us now inquire what correction these formulæ will require when the true elliptical motion is employed.

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* *Monatliche Correspondenz,* XVII. p. 312.

If u is the true anomaly of the earth in the orbit, reckoned from the perihelion, at the time t from the perihelion passage, r the radius vector, a the mean distance of the earth from the sun, or the semi-major axis of the ellipse, we have

$$r = \frac{a(1 - e^2)}{1 + e \cos u}$$

The true direction of the earth's motion at any time is not, as in the circular orbit, at right angles to the direction of the sun, but in that of the tangent to the curve. If we denote the angle which the tangent makes with the radius vector by $90° - i$, we have, by the theory of curves,

$$\cot(90° - i) = \frac{1}{r} \cdot \frac{dr}{du}$$

whence, by the above equation of the ellipse,

$$\tan i = \frac{e \sin u}{1 + e \cos u}$$

and the true *direction* of the earth's motion will be taken into account in our formulæ (675), if for $\odot$ we substitute $\odot - i$.

If v_1 denotes the true velocity of the earth in its orbit at the time t, we have

$$v_1 = r \sec i \frac{du}{dt}$$

and if f is the area described by the radius vector in the time t, F the whole area of the ellipse described in the period T, we have, by KEPLER's first law,

$$\frac{f}{t} = \frac{F}{T}$$

or

$$\frac{df}{dt} = \frac{F}{T}$$

We also have, by the theory of the ellipse,

$$F = \pi a^2 \sqrt{(1 - e^2)}$$

$$\frac{df}{dt} = \frac{r^2}{2} \cdot \frac{du}{dt}$$

and hence

$$\frac{du}{dt} = \frac{2\pi a^2 \sqrt{(1 - e^2)}}{Tr^2}$$

which, substituted in the above value of v_1, together with the value of r, gives

$$v_1 = \frac{a}{\sqrt{(1-e^2)}} \cdot \frac{2\pi}{T} \cdot (1 + e \cos u) \sec i$$

The *mean* value of this velocity is that value which it would have if the small periodic terms depending on u and i were omitted (Art. 361); thus, denoting the mean velocity by v, we have

$$v = \frac{a}{\sqrt{(1-e^2)}} \cdot \frac{2\pi}{T} \tag{679}$$

$$v_1 = v (1 + e \cos u) \sec i \tag{680}$$

If, then, V is the velocity of light, and we put

$$k_1 = \frac{v_1}{V \sin 1''} = k (1 + e \cos u) \sec i$$

we can at once adapt our equations (675) to the case of the elliptical orbit, by introducing k_1 for k and $\odot - i$ for $\odot$, so that we have

$$\lambda' - \lambda = - k (1 + e \cos u) \cos (\odot - \lambda - i) \sec i \sec \beta$$
$$\beta' - \beta = - k (1 + e \cos u) \sin (\odot - \lambda - i) \sec i \sin \beta$$

Developing the sine and cosine of $(\odot - \lambda) - i$, we have

$$\cos (\odot - \lambda - i) \sec i = \cos (\odot - \lambda) + \sin (\odot - \lambda) \tan i$$
$$\sin (\odot - \lambda - i) \sec i = \sin (\odot - \lambda) - \cos (\odot - \lambda) \tan i$$

and substituting the value of $\tan i$, we find

$$\lambda' - \lambda = - k \cos (\odot - \lambda) \sec \beta - ke \cos (\odot - u - \lambda) \sec \beta$$
$$\beta' - \beta = - k \sin (\odot - \lambda) \sin \beta - ke \sin (\odot - u - \lambda) \sin \beta$$

The longitude of the earth's perigee is

$$\Gamma = \odot - u$$

by the introduction of which we have, finally,

$$\left. \begin{aligned} \lambda' - \lambda &= - k \cos (\odot - \lambda) \sec \beta - ke \cos (\Gamma - \lambda) \sec \beta \\ \beta' - \beta &= - k \sin (\odot - \lambda) \sin \beta - ke \sin (\Gamma - \lambda) \sin \beta \end{aligned} \right\} \tag{681}$$

These formulæ differ from (675) only by the second terms, which therefore are the corrections for the eccentricity of the

latitudes and longitudes. The nadir of the point of observation is then to be substituted in the place of the sun:* so that if we put

$\Theta =$ the right ascension of the zenith, or the sidereal time,

the formulæ (675) are rendered immediately applicable to the present case by putting $180° + \Theta, \alpha, \delta$, and k' for $\odot, \lambda, \beta,$ and $k;$ whence we have, *for a point on the terrestrial equator,*

$$\alpha' - \alpha = k' \cos(\Theta - \alpha) \sec \delta$$
$$\delta' - \delta = k' \sin(\Theta - \alpha) \sin \delta$$

Since every point on the surface of the earth moves in a plane parallel to the equator, and this plane is to be regarded as coincident with the plane of the celestial equator, the same formulæ are applicable to every point, provided we introduce into the expression of k' the actual velocity of the point. This velocity varies directly with the circumference of the parallel of latitude, or with its radius; and this radius for the latitude φ is $\rho \cos \varphi'$, φ' being the geocentric latitude and ρ the radius of the earth for that latitude. Hence we have only to put $v'\rho \cos \varphi'$ for v', or $k'\rho \cos \varphi'$ for k', and we obtain for the diurnal aberration in right ascension and declination, for any point of the earth's surface, the formulæ

$$\left. \begin{array}{l} \alpha' - \alpha = k'\rho' \cos \varphi' \cos(\Theta - \alpha) \sec \delta \\ \delta' - \delta = k'\rho' \cos \varphi' \sin(\Theta - \alpha) \sin \delta \end{array} \right\} \quad (684)$$

It only remains to determine k'. For this purpose, we have, by (679),

$$v = \frac{a}{\sqrt{(1 - e^2)}} \cdot \frac{2\pi}{T}$$

which, if T is the length of the sidereal year in sidereal days, gives the value of v for one sidereal day. The motion of a point on the earth's equator in one sidereal day is equal to the circumference of the equator: so that, if a' is the equatorial radius, we have the value of v' referred to the same unit as v, by the formula

$$v' = 2\pi a'$$

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* For the observer is moving directly *from* the west point of his horizon, which is 90° of right ascension in advance of the nadir; and the point from which the earth in its annual revolution is moving is 90° of longitude in advance of the sun.

whence

$$\frac{v'}{v} = \frac{Ta'\sqrt{1-e^2}}{a}$$

But if we put

$p =$ the sun's mean horizontal parallax,

we have

$$\sin p = \frac{a'}{a}$$

and hence we find

$$k' = kT \sin p \sqrt{1-e^2}$$

or, taking STRUVE's value of $k = 20''.4451$, BESSEL's value of $T = 366^d.25637$, ENCKE's value of $p = 8''.57116$, and the eccentricity $e = 0.01677$,

$$k' = 0''.31112$$

This quantity is so small that we may in (684) employ $\cos \varphi$ for $\rho \cos \varphi'$ without sensible error; and hence the diurnal aberration may be found by the formulæ

$$\left. \begin{array}{l} a' - a = 0''.311 \cos \varphi \cos (\Theta - a) \sec \delta \\ \delta' - \delta = 0''.311 \cos \varphi \sin (\Theta - a) \sin \delta \end{array} \right\} \quad (685)$$

The quantity $\Theta - a$ is the hour angle of the star; whence it follows that the diurnal aberration in right ascension for a star on the meridian is $+ 0''.311 \cos \varphi \sec \delta = + 0^s.0207 \cos \varphi \sec \delta$; and the diurnal aberration in declination is then zero.

394. The illustration given in Art. 388 applies also to the diurnal aberration. In one sidereal day each star appears to describe a small ellipse whose major axis is $\sin k' \cos \varphi$, and minor axis $\sin k' \cos \varphi \sin \delta$, the radius of the sphere being unity. For an observer at the pole, where $\cos \varphi = 0$, this ellipse becomes a point, and the diurnal aberration disappears.

395. *The velocity of light.*—The constant k was determined by STRUVE by a comparison of the apparent places of stars at different seasons of the year, and not from the known velocity of light. We can, therefore, determine the velocity of light from this constant. We have, from the preceding articles,

$$V = \frac{v'}{k' \sin 1''} = \frac{v'}{T \sin p \sin k \sqrt{(1-e^2)}}$$

aberration of the fixed stars, and the result is the apparent place at the time t.*

397. If α and δ are the true right ascension and declination of a planet or comet at a time t, α' and δ' the apparent values for the same time, r' its distance from the earth, the mean distance of the earth from the sun being unity, $\Delta\alpha$, $\Delta\delta$, the motion of the planet in right ascension and declination in one mean hour, we have, according to the method II. of the preceding article,

$$\left. \begin{aligned} \alpha' - \alpha &= -r'k''\Delta\alpha \\ \delta' - \delta &= -r'k''\Delta\delta \end{aligned} \right\} \quad (687)$$

in which

$$k'' = \frac{497.78}{3600} \qquad \log k'' = 9.14073$$

These formulæ may also be used for computing the sun's aberration in right ascension and declination, if we take for r' the radius vector of the earth.

HELIOCENTRIC OR ANNUAL PARALLAX OF THE FIXED STARS.

398. The heliocentric parallax of a star is the difference between its true places seen from the earth and from the sun. If the orbit of the earth were a circle with a radius equal to the mean distance from the sun, the maximum difference between the heliocentric and geocentric places of any star would occur when the radius vector of the earth was at right angles to the line drawn from the earth to the star. This maximum corresponds, then, to the horizontal geocentric parallax; and its effect upon the apparent places of stars might be investigated by the methods followed in Chapter IV.; but we prefer to employ here the method just exhibited in the investigation of the aberration, on account of the analogy in the resulting formulæ.

We shall call the maximum angle subtended by the *mean distance* of the earth from the sun, at the distance of the star, the *constant* of annual parallax *of the star*, or, simply, its *annual parallax*. If then we put

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* See GAUSS, *Theoria Motus Corporum Cœlestium*, Art. 71, from which the above article is chiefly extracted. Also, for the application of method III., see the same work, Art. 118, *et seq*.

400. *To find the heliocentric parallax of a star in right ascension and declination, the annual parallax being given.*—By (678), putting pr for k, and $90° + \odot$ for $\odot$, we have, at once,

$$
\left.
\begin{aligned}
a' - a &= - pr \sec \delta \, (\cos \odot \sin a - \sin \odot \cos \epsilon \cos a) \\
\delta' - \delta &= - pr \sin \odot \, (\cos \epsilon \sin \delta \sin a - \sin \epsilon \cos \delta) \\
&\quad - pr \cos \odot \sin \delta \cos a
\end{aligned}
\right\} \quad (691)
$$

401. It can be shown from (690) that, neglecting the small variations produced by the ellipticity of the earth's orbit, the effect of the annual parallax, considered alone, is to cause the star to appear to describe a small ellipse about its mean place in one sidereal year; an effect entirely analogous to that of the annual aberration, Art. 388. But the maximum and minimum of parallax occur when the earth is 90° from the points at which the maximum and minimum of aberration occur: so that the major axes of the parallax and aberration ellipses are at right angles to each other. The combined effect of both aberration and parallax is still to cause the star to describe an ellipse, the major axis of which is equal to the hypothenuse of a right triangle, of which the two legs are respectively equal to the major axes of the two ellipses. For this combined effect is expressed by the following formulæ (taking $r = 1$ for a circular orbit):

$$
(\lambda' - \lambda) = - [k \cos (\odot - \lambda) - p \sin (\odot - \lambda)] \sec \beta
$$
$$
(\beta' - \beta) = - [k \sin (\odot - \lambda) + p \cos (\odot - \lambda)] \sin \beta
$$

which, if we assume c and γ to be determined by the conditions

$$
c \sin \gamma = k \sin \lambda - p \cos \lambda
$$
$$
c \cos \gamma = k \cos \lambda + p \sin \lambda
$$

or

$$
c \sin (\lambda - \gamma) = p
$$
$$
c \cos (\lambda - \gamma) = k
$$

become

$$
(\lambda' - \lambda) = - c \cos (\odot - \gamma) \sec \beta
$$
$$
(\beta' - \beta) = - c \sin (\odot - \gamma) \sin \beta
$$

in which we have $c = \sqrt{(k^2 + p^2)}$. This form for the total effect is entirely analogous to that for the aberration alone.

MEAN AND APPARENT PLACES OF STARS.

402. The formulæ above given enable us to derive the apparent from the mean place, or the mean from the apparent place;

we shall have

$$\alpha' = \alpha + [\tau - i \sin \Omega + i' \sin 2\Omega - i'' \sin 2\mathbb{C} + i''' \sin (\mathbb{C} - \Gamma')$$
$$- i^{iv} \sin 2\odot + i^{v} \sin (\odot - \Gamma) - i^{vi} \sin (\odot + \Gamma)] [m + n \sin \alpha \tan \delta]$$
$$- [9''.2231 \cos \Omega - 0''.0897 \cos 2\Omega + 0''.0886 \cos 2\mathbb{C}$$
$$\quad\quad 9.2240$$
$$\quad\quad + 0''.5510 \cos 2\odot + 0''.0098 \cos (\odot + \Gamma)] \cos \alpha \tan \delta$$
$$- 20''.4451 \cos \epsilon \cos \odot \cos \alpha \sec \delta$$
$$- 20.4451 \sin \odot \sin \alpha \sec \delta$$
$$+ \tau\mu$$
$$- h \sin \Omega + h' \sin 2\Omega - h'' \sin 2\mathbb{C} + h''' \sin (\mathbb{C} - \Gamma')$$
$$\quad\quad - h^{iv} \sin 2\odot + h^{v} \sin (\odot - \Gamma) - h^{vi} \sin (\odot + \Gamma)$$

and

$$\delta' = \delta + [\tau - i \sin \Omega + i' \sin 2\Omega - i'' \sin 2\mathbb{C} + i''' \sin (\mathbb{C} - \Gamma')$$
$$- i^{iv} \sin 2\odot + i^{v} \sin (\odot - \Gamma) - i^{vi} \sin (\mathbb{C} + \Gamma)] n \cos \alpha$$
$$+ [9''.2231 \cos \Omega - 0''.0897 \cos 2\Omega + 0''.0886 \cos 2\mathbb{C}$$
$$\quad\quad 9.2240$$
$$\quad\quad + 0''.5510 \cos 2\odot + 0''.0098 \cos (\odot + \Gamma)] \sin \alpha$$
$$- 20''.4451 \cos \epsilon \cos \odot (\tan \epsilon \cos \delta - \sin \alpha \sin \delta)$$
$$- 20.4451 \sin \odot \cos \alpha \sin \delta$$
$$+ \tau\mu'$$

Putting then, in accordance with the notation adopted in the Nautical Almanac and the American Ephemeris,

$$A = - 20''.4451 \cos \epsilon \cos \odot$$
$$B = - 20.4451 \sin \odot$$
$$C = \quad \tau - i \sin \Omega + i' \sin 2\Omega - i'' \sin 2\mathbb{C} + i''' \sin (\mathbb{C} - \Gamma')$$
$$\quad\quad - i^{iv} \sin 2\odot + i^{v} \sin (\odot - \Gamma) - i^{vi} \sin (\odot + \Gamma)$$
$$D = - 9''.2231 \cos \Omega + 0''.0897 \cos 2\Omega - 0''.0886 \cos 2\mathbb{C}$$
$$\quad\quad 9.2240$$
$$\quad\quad - 0''.5510 \cos 2\odot - 0''.0093 \cos (\odot + \Gamma)$$
$$E = - h \sin \Omega + h' \sin 2\Omega - h'' \sin 2\mathbb{C} + h''' \sin (\mathbb{C} - \Gamma')$$
$$\quad\quad - h^{iv} \sin 2\odot + h^{v} \sin (\odot - \Gamma) - h^{vi} \sin (\odot + \Gamma)$$

which quantities are dependent on the time, and are wholly independent of the star's place; and also

$$a = \cos \alpha \sec \delta \qquad\qquad a' = \tan \epsilon \cos \delta - \sin \alpha \sin \delta$$
$$b = \sin \alpha \sec \delta \qquad\qquad b' = \cos \alpha \sin \delta$$
$$c = m + n \sin \alpha \tan \delta \qquad c' = n \cos \alpha$$
$$d = \cos \alpha \tan \delta \qquad\qquad d' = - \sin \alpha$$

which depend on the star's place, we have

$$\left. \begin{array}{l} a' = a + Aa + Bb + Cc + Dd + E + \tau\mu \\ \delta' = \delta + Aa' + Bb' + Cc' + Dd' + \tau\mu' \end{array} \right\} \quad (692)$$

The logarithms of A, B, C, D are given in the Ephemeris for every day of the year. The residual E never exceeds $0''.05$, and may usually be omitted. The logarithms of $a, b, c, d, a', b', c', d'$ are usually given in the catalogues, but where not given are readily computed by the above formulæ. When the right ascension is expressed in time, the values of a, b, c, d, above given, are to be divided by 15.

403. If we substitute the values of m and n, namely,

for 1800, $m = 46''.0623$ $n = 20''.0607$
      1900, $m = 46\ .0908$ $n = 20\ .0521$

we find the following values of i, i', &c.:

	i	i'	i''	i'''	i^{iv}	i^v	i^{vi}
1800	0.34221	0.00411	0.00405	0.00135	0.02520	0.00254	0.00042
1900	0.34252				0.02521		

	h	h^{iv}	
1800	$+ 0''.052$	$+ 0''.004$	and h', h'', h''', h^v, h^{vi} inappreciable.
1900	$+ 0\ .045$	$+ 0''.003$	

The terms in i^v and i^{vi} in the expression of C may be combined in a single term; for, putting

$$j \cos J = (i^v - i^{vi}) \cos \Gamma$$
$$j \sin J = -(i^v + i^{vi}) \sin \Gamma$$

we have

$$i^v \sin (\odot - \Gamma) - i^{vi} \sin (\odot + \Gamma) = j \sin (\odot + J)$$

and taking for 1800, $\Gamma = 279° 30' 8''$; and for 1900, $\Gamma = 281° 12' 42''$, we find

	j	J
1800	$+ 0.00294$	$83° 10'$
1900	$+ 0.00293$	$81\ \ 55$

from this by the diurnal aberration and the refraction; but the first of these corrections depends on the latitude of the observer and the star's hour angle, and the second upon the star's zenith distance: so that neither of them can be brought into the computation of a star's position until the place of observation and the local time are given.

406. *The fictitious year.*—In the preceding investigations, we have used the expression "beginning of the year," without giving it a definite signification. For the purpose of introducing uniformity and accuracy in the reduction of stars' places, BESSEL proposed a *fictitious year*, to begin at the instant when the sun's mean longitude is 280°. This instant does not correspond to the beginning of the tropical year on the meridian of Greenwich; that is, the (mean) sun is not at this instant on the meridian of Greenwich, but on a meridian whose distance from that of Greenwich can always be determined by allowing for the sun's mean motion. This meridian at which the fictitious year begins will vary in different years; but, since the sun's mean right ascension is equal to his mean longitude (Art. 41), the sidereal time at this meridian when the fictitious year begins is always $18^h 40^m (= 280°)$. By the employment of this epoch, therefore, the reckoning of sidereal time from the beginning of the year is simplified, and, accordingly, it is now generally adopted as the epoch of the catalogues of stars. In the value of log C, which involves the fraction of a year (τ), the same origin of time must be used; and this is attended to in the computation of the Ephemerides, which now give not only the logarithms of A, B, C, and D, but also the value of τ (or its logarithm) reckoned from the beginning of the fictitious year and reduced to decimal parts of the mean tropical year.

For all the purposes of reduction of modern observations, the computer need not enter further into this subject, and may depend upon the Ephemerides.* But, as the subject is inti-

* The reduction of observations made between 1750 and 1850 will be most conveniently performed by the aid of the *Tabulæ Regiomontanæ* of BESSEL. The constants used by BESSEL differ materially from those now adopted in the American and British Almanacs. Professor HUBBARD has given a very simple table by which the values of log A, log B, log C, and log D as given in the *Tab. Reg.* may be reduced to those which follow from the use of PETERS's constants, in the *Astronomical Journal*, Vol. IV. p. 142. The special and general tables for the reduction of stars' places,

is the time in which the sun changes his mean longitude exactly 360°, and is, therefore, found by dividing 360 by the mean daily motion: thus, if we put

$Y = $ the length of the tropical year in mean solar days,

we find

$$Y = 365^d.242220027 - 0^d.00000006886\,t$$

where the value of the second term for $t = 100$ is $0^s.595$, which is the diminution of the length of the tropical year in a century.

The length of the sidereal year is invariable, and is readily found by adding to 365.25 the time required by the sun to move through $22''.617656$ at the rate of his sidereal motion; or, putting

$Y' = $ the length of the sidereal year,

by the proportion

$$360° - 22''.617656 : 360° = 365^d.25 : Y'$$

which gives

$$Y' = 365.256374416 \text{ mean solar days,}$$
$$= 366.256374416 \text{ sidereal days.}$$

408. *The epoch of the sun's mean longitude.*—This term denotes the instant at which the common year begins. The value of the longitude itself at this instant is frequently called "the epoch," and is denoted by E. Its value for January 0 of any year, $1800 + t$, is found by adding the motion in 365 days for each year not a leap year, and the motion in 366 days for each leap year. The motion in 365 days is found from the above value for 365.25 days by deducting one-fourth the mean daily motion, or $14'\,47''.083$: so that, if f denotes the remainder after the division of t by 4, we have, for the epoch of $1800 + t$, Jan. 0, at Paris,

$$E = 279°\,54'\,1''.36 + 27''.605844\,t + 0''.0001221805\,t^2$$
$$- (14'\,47''.083)f \qquad (693)$$

To extend this formula to years preceding 1800, we must put $f - 4$ in the place of f: so that the multiplier of $(- 14'\,47''.083)$ will be, for example, $- 1, - 2, - 3, - 4, - 1$, &c. for the years 1799, '98, '97, '96, '95, &c. But these rules for f will give the

meridian, at the sidereal time $18^h 40^m$ the argument of the table will be the *reduced date;* but at any other sidereal time g the argument must be this reduced date increased by

$$\frac{g - 18^h 40^m}{24^h}$$

which must be always taken < 1 and positive; or by the quantity

$$g' = \frac{g + 5^h 20^m}{24^h}$$

omitting one whole day if $g + 5^h 20^m \gtreqqless 24^h$. Now, in order that the local date may correspond with that supposed in the tables, the day must be supposed to begin at the instant when that point is on the meridian whose right ascension is $18^h 40^m$. Therefore, whenever the right ascension of the sun is as great as $18^h 40^m$, so that the point in question culminates *before* the sun, one day must be added to the common reckoning. Hence the formula for preparing the argument of the tables will be

> *Argument* = Reduced date $+ g' + i$;
> in which we must take $i = 0$ from the beginning of the year to the time when the sun's R. A. $= g$, and $i = + 1$ after this time.

The values of g' are given on p. 16 of the *Tab. Reg.* for given values of g. The values of k are given in Table I.

The values of log A, log B, log C, log D are also given in the Berlin Jahrbuch for the fictitious date; and the constants of precession, nutation, and aberration are the same as those employed by Bessel in the *Tab. Reg.*

411. *Conversion of mean into sidereal time, and vice versa.*—In the explanation of this subject in Chapter II. we said nothing of the effect of nutation, which we will now consider. Let us go back to the definitions and state them more precisely.

1st. The first mean sun, which may be denoted by $\odot_1$, moves uniformly in the ecliptic, returning to the perigee with the true sun. The longitude of this fictitious sun referred to the *mean equinox* is called *the sun's mean longitude.*

2d. The second mean sun, which may be denoted by $\odot_2$, moves

or *vice versa*, according to the rules in Chapter II., employing the V_0 for the fictitious meridian precisely as it was there employed for the meridian of Greenwich.—The longitude of the place to be used here is $k + d$, d being the west longitude of the place from Paris, and k the east longitude of the fictitious meridian from Paris given in Table I.

REDUCTION OF THE APPARENT PLACE OF A PLANET OR COMET.

412. The *observed* place of a planet (or comet) being freed from the effect of refraction, diurnal aberration, and geocentric parallax, we have the apparent geocentric place, referred to the true equator and equinox of the time of observation, and affected by the planetary aberration. For the calculation of a planet's orbit from three or more observations at different times, it is necessary to refer its places at these times to the same common fixed planes, which is most readily effected by reducing all the places to the equinox of the beginning of the year in which the observations are made, or, when the observations extend beyond one year, to the beginning of any assumed year. To effect this, we must apply to each apparent geocentric place—1st. The aberration (687), with its sign reversed, in computing which the position of the observer on the surface of the earth may be considered by taking r' equal to the actual distance of the planet from the observer at the time of observation. This distance is found from the geocentric distance at the same time with the parallax, by the equation (137).

2d. The nutation for the date of the observation, with its sign reversed.

3d. The precession from the date of the observation to the assumed epoch, which will be subtracted or added according as the epoch precedes or follows the date.

But the nutation and precession are most conveniently computed together by the aid of the constants C and D used for the fixed stars. These constants being taken for the date, c, d, c', and d' are to be computed as in Art. 402, with the right ascension and declination of the planet; and then to the α and δ, already corrected for aberration, we apply the corrections $- (Cc + Dd)$ and $- (Cc' + Dd')$ respectively. The place thus obtained is the *true place of the planet referred to the mean equinox of the beginning of the year.* If the several observations are in different

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415. Now, the sun's declination is equal to the obliquity only when it has reached its maximum (northern or southern) limit, that is, precisely at the solstitial points. But, since the sun will, in general, not arrive at the solstice at the same time that it culminates at the particular meridian at which the observation is made, we cannot directly measure this maximum by meridian observations. But we can measure the declination at several successive transits near the solstice, and then by interpolation infer the maximum value. A simpler practical process (which we shall explain fully below) is to reduce each observation to the solstice; but this requires us to know (at least approximately) the *time* when the sun arrives at the solstice, and this, again, supposes a knowledge of the position of the equinoctial points, which are 90° distant from the solstitial points.

The position of the equinoctial points may be determined by observing the sun's declination on several successive days near the time of the equinoxes, and, by interpolation, finding the time when the declination is zero. At the same time, a comparison must be made between the times of transit of the sun and some star, adopted as a *fundamental* star: so that the distance of the star from the equinoctial point, or its right ascension, is fixed. We may then regard the star as a fixed point of comparison by which the instants when the sun arrives at any given points (as the solstices) may be determined. But, instead of finding the equinoctial points by a direct interpolation, it is preferable in this case also to refer each observation to the equinox, which, as will be seen below, requires an approximate knowledge of the obliquity of the ecliptic.

The determination of these two elements, the obliquity of the ecliptic and the position of the equinoctial points, is, therefore, effected by successive approximations; but, in the actual state of astronomy, the approximations are already so far carried out that the remaining error in either element can be treated as a differential which, by a judicious arrangement of the observations, produces only insensible errors of a higher order in the other element. I proceed to treat fully of the precise practical methods.

416. *Determination of the obliquity of the ecliptic.*—Let D be the sun's apparent declination derived from an observation near the solstice; A its apparent right ascension at the time of the obser-

vation, derived from the solar tables; ϵ the apparent obliquity of the ecliptic for the same time. If the sun were exactly in the ecliptic, we should have, by (34),

$$\sin A \tan \epsilon = \tan D$$

but accuracy requires that the sun's latitude, β, should be taken into account. We have, by (29),

$$\tan D - \tan \epsilon \sin A = \frac{\sin \beta}{\cos D \cos \epsilon}$$

which, if we put

$$\tan D' = \tan \epsilon \sin A \qquad (694)$$

becomes

$$\tan D - \tan D' = \frac{\sin (D - D')}{\cos D \cos D'} = \frac{\sin \beta}{\cos D \cos \epsilon}$$

whence, with sufficient accuracy, since β never exceeds $1''$,

$$D - D' = \beta \sec \epsilon \cos D \qquad (695)$$

Hence, if the correction $\beta \sec \epsilon \cos D$ is subtracted from the given declination D, we shall obtain the *reduced* declination D', from which, by (694), we can deduce ϵ. It is evident that D' is the declination of the point in which the ecliptic is intersected by the declination circle drawn through the sun's centre, and we may call the quantity $\beta \sec \epsilon \cos D$ the *reduction to the ecliptic*. Near the solstices, however, this reduction does not sensibly differ from β, since $\cos \epsilon$ and $\cos D$ are then very nearly equal. We shall, therefore, in the present problem, find the reduced declination by the formula $D' = D - \beta$; and then we have, by (694),

$$\tan \epsilon = \tan D' \operatorname{cosec} A \qquad (696)$$

Instead of computing ϵ from this equation directly, it is usual to employ its development in series by which the difference of ϵ and D' is obtained. For, since A near the northern solstice is nearly $90°$, if we put

$$u = 90° - A$$

u will be a small angle whose cosine and secant will not differ much from unity, and the equation (696), expressed in the form

$\tan D' = \tan \epsilon \cos u$, will be developed in the series [Pl. Trig., Art. 254]

$$D' - \epsilon = q \sin 2\epsilon + \tfrac{1}{2} q^2 \sin 4\epsilon + \tfrac{1}{3} q^3 \sin 6\epsilon + \&c.$$

in which

$$q = \frac{\cos u - 1}{\cos u + 1} = - \tan^2 \tfrac{1}{2} u$$

and the terms of the series are expressed in arc.   Reducing to seconds, and putting

$$x = \text{the reduction to the solstice,}$$

or

$$x = \frac{\tan^2 \tfrac{1}{2} u}{\sin 1''} \sin 2\epsilon - \frac{\tan^4 \tfrac{1}{2} u}{\sin 1''} \sin 4\epsilon + \&c. \tag{697}$$

we have, *at the northern solstice,*

$$\epsilon = D' + x = D - \beta + x \tag{698}$$

The reduction x can be tabulated, for any assumed value of ϵ, with the argument u.   The changes of the tabular numbers depending on a change of the obliquity may also be given in the table: so that these numbers may be readily made to correspond to any assumed obliquity.

For the southern solstice, we take $u = 270° - A$, and the equation (696) will give $\tan D' = - \tan \epsilon \cos u$, the development of which gives the algebraic sum $D' + \epsilon$; but we can avoid the use of two formulæ by throwing this change of sign upon ϵ, regarding the obliquity obtained from the southern solstice as negative, during the computation.   This simply changes the sign of the reduction x.

417. Let us now inquire what effect an error in the right ascensions taken from the tables, or in u, will produce in the computed value of ϵ.   Differentiating the equation (696) with reference to ϵ and $A = \pm 90° - u$, we find

$$d\epsilon = \tfrac{1}{2} \tan u \sin 2\epsilon \, du$$

If we suppose the error in the tabular right ascension of the sun to be in any case as great as one second of time (the actual probable error, however, being much less), and, therefore, substitute in this equation $du = 15''$, $\epsilon = 23° 27'.5$, we find

$$d\epsilon = 5''.48 \tan u$$

following apparent declinations of the sun's centre, observed at the Washington Observatory by Professor COFFIN and Lieutenant PAGE, with the mural circle.

1846.	D	1846.	D
June 16	23° 21′ 56″.02	December 14	— 23° 14′ 17″.26
" 19	26 28 .19	" 15	17 33 .82
" 20	27 6 .79	" 16	20 22 .94
" 23	26 39 .92	" 18	24 32 .69
" 27	20 17 .84	" 21	27 20 .43
		" 22	27 19 .64
		" 23	26 49 .82
		" 29	14 1 .20

Taking $5^h 8^m 11^s.2$ as the longitude of Washington from Greenwich, we find, for apparent noon at Washington, the following values of the sun's right ascension and latitude from the Nautical Almanac:

1846.	A	β	1846.	A	β
June 16	$5^h 38^m 37^s.13$	+ 0″.18	December 14	$17^h 26^m 52^s.78$	+ 0″.35
" 19	5 51 5 .77	— 0 .19	" 15	17 31 18 .48	+ 0 .46
" 20	5 55 15 .44	— 0 .82	" 16	17 35 44 .38	+ 0 .57
" 23	6 7 44 .44	— 0 .63	" 18	17 44 36 .91	+ 0 .72
" 27	6 24 22 .00	— 0 .72	" 21	17 57 56 .69	+ 0 .70
			" 22	18 2 23 .39	+ 0 .64
			" 23	18 6 50 .09	+ 0 .50
			" 29	18 33 28 .11	— 0 .19

Supposing no tables of the reduction at hand, let us first reduce the observations at the summer solstice by the original equation (696). Subtracting β from the observed values of D, we then have

	D′	log tan D′	log cosec A	log tan ε	ε
June 16	23° 21′ 55″.84	9.6355081	0.0018927	9.6374008	23° 27′ 23″.61
" 19	26 28 .38	.6370823	03278	4101	25 .22
" 20	27 7 .11	.6373056	00930	3986	23 .23
" 23	26 40 .55	.6371524	02478	4002	23 .51
" 27	20 18 .56	.6349452	24592	4044	24 .25

Apparent obliquity = 23 27 23 .96

eliminate the error of the assumed latitude of Washington,*
we have, finally,

Mean obliquity for 1846.0 from observation = 23° 27′ 33″.03
The same by PETERS's formula (646) with ⎫
   the annual decrease 0″.4645 ⎭ = "   " 32 .85

420. The secular variation of the obliquity is found by com-
paring its values at very distant epochs. The observations of
BRADLEY from 1753 to 1760 gave for 1757.295 the mean obliquity
23° 28′ 14″.055. The observations at the Dorpat Observatory
gave for 1825.0 the mean obliquity 23° 27′ 42″.607. Hence

$$\text{Annual var.} = -\frac{31''.448}{67.705} = -0''.4645$$

BESSEL found — 0″.457 by comparing BRADLEY's observations
with his own.

The secular variation is also found in Physical Astronomy,
theoretically. The value thus obtained by PETERS in his *Nume-
rus Constans Nutationis* is — 0″.4738, as given in the formulæ
(646).

421. *Determination of the equinoctial points, and the absolute right
ascension and declination of the fixed stars.*—The declinations of the
fixed stars are either directly measured by the fixed instruments
of the observatory, or deduced immediately from their observed
meridian zenith distances (corrected for refraction) by the formula
$\delta = \varphi - \zeta$. The practical details, which depend on the instru-
ment employed, will be given in Vol. II. Here we have only
to observe that the immediate result of such a measurement is
the *apparent* declination at the time of observation, which must
then be reduced to the mean declination for some assumed
epoch by the formulæ of the preceding chapter.

The position of the equinoctial points is determined as soon
as we have found the right ascension of *one* fixed star; and this
is done by deducing from observation the *difference* between the

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* The latitude employed in deducing the declinations was 38° 53′ 39″.25. The
latitude given by the culminations of *Polaris* is 38° 53′ 39″.52 (*Washington Astr.
Obs.*, Vol. I., App. p. 113). If we adopt the latter value, the obliquity derived from
the northern solstice will be increased by 0″.27, and that derived from the southern
solstice will be diminished by the same quantity; and the difference then remaining
between the two results will be only 0″.67.

$$dA = - d\epsilon \frac{2 \tan A}{\sin 2\epsilon}$$

The correction of the declination D' is composed of the corrections in the latitude φ, and the zenith distance ζ; since, by the formula $D = \varphi - \zeta$, we have

$$dD = d\varphi - d\zeta$$

But $d\zeta$ is itself composed of the corrections required in the refraction and the sun's parallax and the correction for any error peculiar to the zenith distance ζ, which affects the meridian instrument employed in the observation. Denoting the correction of the refraction by dr, that of the sun's parallax by $dp \sin \zeta$, that of the instrument for the zenith distance ζ by $f(\zeta)$, we have

$$dD = d\varphi - [dr - dp \sin \zeta + f(\zeta)]$$

The effect of this correction upon A is found, by differentiating (699) with reference to D' (regarding dD as equal to dD'), to be

$$dA = dD \frac{2 \tan A}{\sin 2D'}$$

If then α' denotes the corrected mean right ascension of the star, free from all constant errors, we have

$$\alpha' = \alpha + \left[d\varphi - dr + dp \sin \zeta - f(\zeta)\right] \frac{2 \tan A}{\sin 2D'} - d\epsilon \frac{2 \tan A}{\sin 2\epsilon}$$

This formula shows that nearly all the errors will be eliminated by taking the mean between two observations taken at the same zenith distance (or the same declination), the one near the vernal, the other near the autumnal equinox. For, the first observation being taken when the declination is D' and right ascension A, at the second one the same declination D' will give the right ascension $180° - A$, the tangent of which is the negative of that of A. The temperature being generally different at the two seasons of the year, we cannot assume that the error in the refraction tables will be the same at both observations unless we can also assume that the *law* of correction of the refraction for temperature is perfectly known. So, also, we must admit the possibility that such changes of temperature change the instrumental correction; but the corrections of the latitude and the parallax will remain the same. Hence, if α_1 is the mean right

Feb. 23. $D = -$ 9° 46' 15".85	‖	Oct. 17. $D = -$ 9° 17' 53".12
" " $T =$ 22ʰ 26ᵐ 28ˢ.11	‖	" " $T =$ 13ʰ 28ᵐ 40ˢ.01
" " $t =$ 0 5 18 .99	‖	" 16. $t =$ 0 5 22 .97

The times of transit are corrected for the supposed error and rate of the clock.

For the dates of the two observations, the apparent obliquity of the ecliptic and the sun's latitude are as follows:

	Feb. 23.	Oct. 17.
ϵ	23° 27' 26".10	23° 27' 24".35
β	+ 0 .33	− 0 .13

whence

	Feb. 23.	Oct. 17.
$-\beta \sec \epsilon \cos D$	− 0 .35	+ 0 .14
$D' -$	9 46 16 .20	− 9 17 52 .98
$\log \tan D'$	n9.236063	n9.214105
$\log \cot \epsilon$	0.362585	0.362595
$\log \sin A$	n9.598648	n9.576700
A	22ʰ 26ᵐ 28ˢ.17	13ʰ 28ᵐ 40ˢ.14
$A - T$	+ 0.06	+ 0.13
$t + A - T = \alpha$	0 5 19.05	0 5 23 .10
Reduction to 1850.0	+ 12 .15	+ 8 .12
Mean α for 1850.0 $=$	0 5 31 .20	0 5 31 .22

The reduction to 1850 is here used because it can be taken directly from the general tables for reducing the apparent places of stars to mean places, given in the volume of Washington Observations for 1847. Taking the mean of the two observations, we have, finally,

Mean R. A. of γ Pegasi for 1850.0 $=$ 0ʰ 5ᵐ 31ˢ.21

422. When, by the combination of a great number of observations, the right ascension of a fundamental star is thus established, the right ascensions of all other stars follow from the differences of time between their several transits and that of the fundamental star. But, in the present state of the star catalogues, it will be preferable not to limit the object of these observations to determining a single star. The constant use of the same fundamental stars as "clock stars" (stars near the equator by which the clock correction and rate are found) gives to the *relative* right ascensions of these stars (as derived from all their observed transits during one or more years) a high degree of accuracy. Assuming,

CHAPTER XIII.

DETERMINATION OF ASTRONOMICAL CONSTANTS BY OBSERVATION.

423. I SHALL not attempt to enter into all the details of the methods by which the various astronomical constants are determined from observations, but shall confine myself to a sketch of their general principles, which will serve as an introduction to the special papers to be found in astronomical memoirs and other sources.

THE CONSTANTS OF REFRACTION.

424. The general refraction formula (191) involves the two constants α and β, both of which may be found from theory by the formulæ (178) and (176). But, as the refraction formula was deduced from an hypothesis, it was not to be expected that the theoretical values of α and β would give refractions in entire accordance with observation. The discrepancies, however, are exceedingly small: so small, indeed, that the formula may be regarded as representing well enough the *law* of refraction, without resorting to any new hypothesis; and to perfect it we have only to give the constants slightly amended values, whereby the computed refractions are made to harmonize entirely with those deduced from observation. To deduce the corrections of α and β, we can employ the concise expression of the refraction (213), or

$$(1 - \alpha) \, r = \sin^2 z \, \sqrt{\frac{2}{\beta}} \cdot Q$$

The factor $1 - \alpha$ differs so little from unity that we may regard it as constant in determining the small correction of r, and, therefore, by differentiating, we have

$$(1 - \alpha) \, dr = \sin^2 z \, \sqrt{\frac{2}{\beta}} \left[\frac{dQ}{d\alpha} \cdot d\alpha + \left(\frac{dQ}{d\beta} - \frac{Q}{2\beta} \right) d\beta \right]$$

By (217) and (210) we have

$$\frac{dQ}{d\alpha} = \frac{dQ}{dx} \cdot \frac{dx}{d\alpha} = \frac{1 - x}{x} \cdot \frac{\beta}{\sin^2 z} \, Q'$$

The true zenith distances which would be obtained by a table of refractions founded on the corrected constants will be $z + dr$ and $z_1 + dr_1$; and, therefore, if $d\varphi$ denotes the correction of the assumed latitude, we shall have

$$90^\circ - (\varphi + d\varphi) = z + dr + 90^\circ - \delta$$
$$90^\circ - (\varphi + d\varphi) = z_1 + dr_1 - (90^\circ - \delta_1)$$

whence, by taking the mean,

$$90^\circ - \varphi - d\varphi = \tfrac{1}{2}(z + z_1) + \tfrac{1}{2}(\delta_1 - \delta) + \tfrac{1}{2}(dr + dr_1)$$

The quantity $\delta_1 - \delta$ is merely the very small change of the star's declination between the two culminations, arising from precession and nutation, which is accurately known. If we substitute the values of dr and dr_1 in terms of $d\alpha$ and $d\beta$, and then put

$$a = \tfrac{1}{2}(A + A_1) \qquad\qquad b = \tfrac{1}{2}(B + B_1)$$
$$n = \tfrac{1}{2}(z + z_1) + \tfrac{1}{2}(\delta_1 - \delta) + \varphi - 90^\circ$$

we have the equation of condition

$$d\varphi + a\, d\alpha_0 + b\, d\beta_0 + n = 0 \qquad\qquad (702)$$

By employing a number of stars which culminate at various zenith distances, we shall obtain a number of such equations, in which the coefficients a and b will have different values: so that the solution of all these equations by the method of least squares will determine the three unknown quantities $d\varphi$, $d\alpha_0$, and $d\beta_0$.

THE CONSTANT OF SOLAR PARALLAX.

425. *The constant of solar parallax is the sun's mean equatorial horizontal parallax*, or its horizontal parallax when its distance from the earth is equal to the semi-major axis of the earth's orbit. The constant of parallax of any planet is also its parallax when its distance from the earth is equal to the semi-major axis of the earth's orbit: so that the constant of solar parallax belongs to the whole solar system.

The *relative* dimensions of the orbits of the planets are known from the periodic times of their revolutions about the sun, since, by KEPLER's third law, the squares of their periodic times are proportional to the cubes of their mean distances from the sun,

and therefore

$$\sin \pi = \frac{\varDelta_0}{\varDelta} \sin \pi_0 \qquad\qquad \sin \varkappa_1 = \frac{\varDelta_0}{\varDelta_1} \sin \pi_0$$

The quantities $\varDelta$ and $\varDelta_1$ are to be found from the planetary tables, or directly from the Nautical Almanac, where they are expressed in terms of $\varDelta_0$ as the unit: so that their values there given are the values of the ratios $\frac{\varDelta}{\varDelta_0}$ and $\frac{\varDelta_1}{\varDelta_0}$. Hence we shall put $\varDelta_0 = 1$ in the preceding formulæ, and also put the arcs for their sines (since the greatest planetary parallax is only $35''$): so that we have

$$\pi = \frac{\pi_0}{\varDelta} \qquad\qquad \pi_1 = \frac{\pi_0}{\varDelta_1}$$

Then, by (114),

$$p = \rho \pi \sin \left[\zeta' - (\varphi - \varphi') \right] = \frac{\rho \pi_0}{\varDelta} \sin \left[\zeta' - (\varphi - \varphi') \right]$$

$$p_1 = \rho_1 \pi_1 \sin \left[\zeta_1' - (\varphi_1 - \varphi_1') \right] = \frac{\rho_1 \pi_0}{\varDelta_1} \sin \left[\zeta_1' - (\varphi_1 - \varphi_1') \right]$$

But we also have

$$\zeta = \varphi - \delta \qquad\qquad \zeta_1 = \varphi_1 - \delta_1$$

and hence

$$\zeta - \zeta_1 = (\zeta' - p) - (\zeta_1' - p_1) = \varphi - \varphi_1 - (\delta - \delta_1)$$

from which we obtain

$$p - p_1 = \zeta' - \zeta_1' - (\varphi - \varphi_1) + (\delta - \delta_1)$$

As the small difference $\delta - \delta_1$ will be accurately known, the observations being taken nearly on the same meridian, all the quantities in the second member of this equation may be regarded as known. Hence, putting

$$\left. \begin{array}{l} n = \zeta' - \zeta_1' - (\varphi - \varphi_1) + (\delta - \delta_1) \\[2ex] a = \frac{\rho}{\varDelta} \sin \left[\zeta' - (\varphi - \varphi') \right] - \frac{\rho_1}{\varDelta_1} \sin \left[\zeta_1' - (\varphi_1 - \varphi_1') \right] \end{array} \right\} \quad (703)$$

we obtain the equation

$$a \pi_0 = n \qquad\qquad (704)$$

which determines π_0. If the zeniths of the two places of observation are on opposite sides of the star (which is the most favor-

where $\Delta\delta$ and $\Delta\delta_1$ are in each case the planet's declination *minus* the star's declination, and their signs are to be carefully observed. For computing the coefficient a, the apparent zenith distances will be obtained by the formulæ

$$\zeta' = \varphi - (D + \Delta\delta) \qquad\qquad \zeta_1' = \varphi_1 - (D + \Delta\delta_1)$$

so that we have

$$a = \frac{\rho}{\varDelta} \sin\left[\varphi' - (D + \Delta\delta)\right] - \frac{\rho_1}{\varDelta_1} \sin\left[\varphi_1' - (D + \Delta\delta_1)\right] \quad (706)$$

and then, as before,

$$a\pi_0 = n$$

A great number of such corresponding observations will be necessary in order to determine π_0 with accuracy; and all the equations of the form just given are to be combined by the method of least squares. Thus, from the equations

$$a\pi_0 = n, \qquad\qquad a'\pi_0 = n', \qquad\qquad a''\pi_0 = n'', \&c.$$

we obtain the final equation

$$[aa]\,\pi_0 = [an] \qquad\text{or}\qquad \pi_0 = \frac{[an]}{[aa]}$$

in which $[aa] = aa + a'a' + a''a'' + \&c.$, and $[an] = an + a'n' + a''n'' + \&c.$

427. *To find the solar parallax by extra-meridian observations of a planet.*—The preceding process will require but a slight modification. The difference of apparent declination of the planet and a neighboring star is measured at both stations with a micrometer attached to an equatorial telescope, and is to be corrected for refraction. The quantity n will then be found by (705). The coefficient a will now be the difference of the coefficients of parallax in declination, computed by the formulæ (143), according to which, if we put

$$\tan\gamma = \frac{\tan\varphi'}{\cos(\Theta - a)} \qquad\qquad \tan\gamma_1 = \frac{\tan\varphi_1'}{\cos(\Theta_1 - a_1)}$$

we shall have

$$a = \frac{\rho}{\Delta} \cdot \frac{\sin \varphi' \sin (\gamma - \delta)}{\sin \gamma} - \frac{\rho_1}{\Delta_1} \cdot \frac{\sin \varphi_1' \sin (\gamma_1 - \delta_1)}{\sin \gamma_1} \qquad (707)$$

in which Θ and Θ_1 are the local sidereal times of the observations, α and α_1 the right ascensions, δ and δ_1 the declinations of the planet at these times. The equation of condition from each pair of corresponding observations of the same star will then be, as before, $a \pi_0 = n$.

If several comparisons are made at either place on the same day, these must first be combined, and reduced, as it were, to a single comparison. Thus, if we put

$$c = \frac{\rho}{\Delta} \cdot \frac{\sin \varphi' \sin (\gamma - \delta)}{\sin \gamma}$$

we have, for each comparison of the planet with the star,

$$\delta = D + \Delta\delta + c\pi_0$$

and if m such comparisons are made, their mean will be

$$\delta = D + \frac{1}{m} \Sigma(\Delta\delta) + \pi_0 \cdot \frac{\Sigma(c)}{m}$$

In like manner, at the second place, we shall have for m_1 observations the equation

$$\delta_1 = D + \frac{1}{m_1} \Sigma(\Delta\delta_1) + \pi_0 \frac{\Sigma(c_1)}{m_1}$$

and, taking the difference of these equations, we shall put

$$n = \delta - \delta_1 - \left(\frac{\Sigma(\Delta\delta)}{m} - \frac{\Sigma(\Delta\delta_1)}{m_1} \right)$$

$$a = \frac{\Sigma(c)}{m} - \frac{\Sigma(c_1)}{m_1}$$

The equation of condition $a \pi_0 = n$ will then represent all the observations on the same day at the two places.

428. The equations of condition will involve smaller numbers and be more easily solved if the unknown quantity is, not the whole parallax, but the correction of some assumed value of the parallax not greatly in error. In this case we may correct each observed difference $\Delta\delta$ for parallax, employing the assumed value

of π_0; and, proceeding as before, we shall have the equation of condition $a \Delta\pi_0 = n$, in which $\Delta\pi_0$ is the required correction of π_0.

429. If but one limb of the planet is observed at one or both the stations, it will be necessary to introduce the correction for the semidiameter. As the semidiameter itself should then be regarded as an unknown quantity, to be found if possible from the observations, its complete expression, in terms of all the corrections which the observations may require, is to be employed. This will be found in Article 435.

430. The differences of right ascension of the planet and a neighboring star may also be employed in the same manner as the differences of declination, the places of observation being in that case in widely different longitudes. We have only to introduce into (707) the coefficients of the parallax in right ascension computed by the first equation of (143), and in the expression of n substitute right ascensions for declinations.

431. The only planets which are near enough to the earth for the successful application of this method are Mars and Venus.

Mars is nearest to the earth at the time of opposition, and for this time the British Nautical Almanac furnishes an Ephemeris of stars to be observed with the planet. All the oppositions, however, are not equally favorable. The mean distance of Mars from the sun being $= 1.524$, and the eccentricity of the orbit $= 0.0933$, while the mean distance of the earth $= 1$ and the eccentricity of its orbit $= 0.017$, it follows that for an opposition in which Mars is at its perihelion while the earth is at its aphelion, the distance of the two bodies will be 0.365; but for one in which Mars is at its aphelion and the earth at its perihelion, their distance will be 0.683. Thus the former case will be nearly twice as favorable as the latter.

Venus is nearest to the earth at the time of inferior conjunction, but at that time can very rarely be compared micrometrically with stars, as the observations would be made with the sun above the horizon. The most favorable position of this planet is at or near its stationary points, where the changes of the planet's place are small and may therefore be accurately computed, while the distance from the earth is still not too great.*

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* GERLING, *Astron. Nach.*, No. 599.

δ, δ_1 = the apparent declinations of the limb observed at the north and south stations respectively,

D, D_1 = the geocentric declinations of the moon's centre at the respective times of observation,

φ, φ_1 = the geographical latitudes of the stations,

γ, γ_1 = the reductions of the latitudes for the earth's compression,

ρ, ρ_1 = the distances of the stations from the earth's centre, the equatorial radius being unity,

P, P_1 = the moon's horizontal parallax at the times of the observation, respectively;

then, the apparent zenith distance of the limb and the geocentric zenith distance of the centre of the moon being, for the northern station,

$$\zeta' = \varphi - \delta \qquad \text{and} \qquad \zeta = \varphi - D$$

we have, by (255),

$$\sin (D - \delta) = [\rho \sin (\zeta' - \gamma) \mp k] \sin P$$

where k is the constant ratio of the radii of the moon and the earth, for which the value 0.272956 may be assumed; and the upper or lower sign of k is to be used according as the upper or lower limb is observed.

At the southern station we have

$$\zeta_1' = \delta_1 - \varphi_1 \qquad\qquad \zeta_1 = D_1 - \varphi_1$$

and hence, taking the reduction γ_1 as a positive quantity,

$$\sin (D_1 - \delta_1) = - [\rho_1 \sin (\zeta_1' - \gamma_1) \pm k] \sin P_1$$

where the sign of k is reversed, since the same limb will be an upper limb at one station and a lower limb at the other. For brevity, put

$$m = \rho \sin (\zeta' - \gamma) \mp k$$
$$m_1 = \rho_1 \sin (\zeta_1 - \gamma_1) \pm k$$

then, from the equations

$$\sin (D - \delta) = m \sin P \qquad\qquad \sin (D_1 - \delta_1) = - m_1 \sin P_1$$

we derive,* neglecting powers of sin P above the third,

$$D - \delta = \frac{m \sin P}{\sin 1''} + \tfrac{1}{6} \cdot \frac{m^3 \sin^3 P}{\sin 1''}$$

$$D_1 - \delta_1 = -\frac{m_1 \sin P_1}{\sin 1''} - \tfrac{1}{6} \cdot \frac{m_1^3 \sin^3 P_1}{\sin 1''}$$

If now the times of the two observations reckoned at the same first meridian are T and T_1, and for the middle time $t = \tfrac{1}{2}(T + T_1)$ we deduce from the lunar tables the hourly increase of the moon's declination, or $\dfrac{dD}{dt}$, we shall have, with regard to second differences,

$$D_1 - D = (T_1 - T)\frac{dD}{dt}$$

Again, if we denote the moon's horizontal parallax at the time t by p, and compute from the tables its hourly increase for this time, or $\dfrac{dp}{dt}$, we shall have

$$\sin P = \sin p + \cos p \sin 1'' (T - t)\frac{dp}{dt}$$

$$\sin P_1 = \sin p + \cos p \sin 1'' (T_1 - t)\frac{dp}{dt}$$

Taking the difference of the above values of $D - \delta$ and $D_1 - \delta_1$, we obtain, therefore,

$$0 = \left[(T_1 - T)\frac{dD}{dt} - (\delta_1 - \delta)\right] \sin 1'' + (m^3 + m_1^3)\frac{\sin^3 p}{6}$$

$$+ \cos p \sin 1'' \frac{dp}{dt}\left[m(T - t) + m_1(T_1 - t)\right]$$

$$+ (m + m_1)\sin p \qquad\qquad (708)$$

The parallax is sufficiently well known for the accurate computation of the terms in $\sin^3 p$ and $\dfrac{dp}{dt}$: so that the only unknown quantity in this equation is the last term. In this term we have

$$m + m_1 = \rho \sin(\zeta' - \gamma) + \rho_1 \sin(\zeta_1' - \gamma_1) \qquad (709)$$

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* By the formula, [Pl. Trig. (413)],

$$x = \sin x + \tfrac{1}{6}\sin^3 x + \&c.$$

where the second member is to be reduced to seconds by dividing it by sin 1''.

may isolate the terms which involve it, as follows. Neglecting the fourth powers of the eccentricity e, we have, by (84) and (83),

$$\rho = 1 - \tfrac{1}{2} e^2 \sin^2 \varphi$$

$$\gamma = \frac{e^2 \sin 2 \varphi}{2 \sin 1''}$$

But when we neglect the fourth powers of e, or the square of the compression c, we have, by (81),

$$c = \tfrac{1}{2} e^2$$

by which we obtain the somewhat simpler forms,

$$\rho = 1 - c \sin^2 \varphi$$

$$\gamma = \frac{c \sin 2 \varphi}{\sin 1''}$$

These values substituted in m give, by neglecting the square of c,

$$m = (1 - c \sin^2 \varphi) \sin \left(\zeta' - \frac{c \sin 2 \varphi}{\sin 1''} \right) \mp k$$

$$= (1 - c \sin^2 \varphi) (\sin \zeta' - c \sin 2 \varphi \cos \zeta') \mp k$$

$$= \sin \zeta' - c (\sin^2 \varphi \sin \zeta' + \sin 2 \varphi \cos \zeta') \mp k$$

and, similarly,

$$m_1 = \sin \zeta_1' - c (\sin^2 \varphi_1 \sin \zeta_1' + \sin 2 \varphi_1 \cos \zeta_1') \pm k$$

The effect of the compression will be insensible in the terms involving $\sin^3 p$, in which we may take

$$m^2 = (\sin \zeta' \mp k)^2 \qquad\qquad m_1^2 = (\sin \zeta_1' \pm k)^2$$

and the same approximation is allowable in the term in $\dfrac{dp}{dt}$. If then we make these substitutions in (708), we obtain the following expanded equation:

$$0 = \left[(T_1 - T) \frac{dD}{dt} - (\delta_1 - \delta) \right] \sin 1'' + \left[(\sin \zeta' \mp k)^2 + (\sin \zeta_1' \mp k)^2 \right] \frac{\sin^3 p}{6}$$

$$+ \cos p \frac{dp}{dt} \sin 1'' \left[(\sin \zeta' \mp k) (T - t) + (\sin \zeta_1' \pm k) (T_1 - t) \right]$$

$$+ \mu \sin p_0 (\sin \zeta' + \sin \zeta_1')$$

$$- c \mu \sin p_0 [\sin^2 \phi \sin \zeta' + \sin 2 \phi \cos \zeta' + \sin^2 \phi_1 \sin \zeta_1' + \sin 2 \phi_1 \cos \zeta_1']$$

If this equation be divided by μ, it may be expressed under the form

$$0 = n + x\,(a - cb) \tag{713}$$

where the notation is as follows:

$$n = \frac{\sin 1''}{\mu}\left[(T_1 - T)\frac{dD}{dt} - (\delta_1 - \delta)\right]$$

$$+ \frac{\sin^3 p}{6\,\mu}\left[(\sin \zeta' \mp k)^3 + (\sin \zeta_1' \pm k)^3\right]$$

$$+ \frac{\sin 1''}{\mu}\left[(\sin \zeta' \mp k)(T - t) + (\sin \zeta_1 \pm k)(T_1 - t)\right]\frac{dp}{dt}\cos p$$

$$a = \sin \zeta' + \sin \zeta_1'$$

$$b = \sin^2\varphi \sin \zeta' + \sin 2\varphi \cos \zeta' + \sin^2 \varphi_1 \sin \zeta_1' + \sin 2\varphi_1 \cos \zeta_1'$$

$$x = \sin p_0$$

It is here to be observed that we have taken γ_1 as a positive quantity even for the southern station: so that $\sin 2\varphi_1$ must be taken positively in computing b.

Let us now suppose we have obtained from a large number of such corresponding observations the equations

$$0 = n\ + x\,(a\ - cb)$$
$$0 = n' + x\,(a' - cb')$$
$$0 = n'' + x\,(a'' - cb'')$$
$$\&c.$$

Multiplying these respectively by a, a', a'', &c., and then forming their sum, we have

$$0 = [an] + [aa]\,x - [ab]\,cx$$

where $[an] = an + a'n' + \&c.$, $[aa] = aa + a'a' + \&c.$, &c. The last term is very small: so that an approximate value of x may be found by neglecting it, whence

$$(x) = -\frac{[an]}{[aa]}$$

which value may then be employed with sufficient accuracy in the term $[ab]cx$; we thus find the complete value

$$x = -\frac{[an]}{[aa]} + \frac{[an]}{[aa]}\frac{[ab]}{[aa]}\cdot c \tag{714}$$

THE MEAN SEMIDIAMETERS OF THE PLANETS.

435. The apparent equatorial semidiameter of a planet when its distance from the earth is equal to the earth's mean distance from the sun is the constant from which its apparent semidiameter at any other distance can be found by the formula

$$s = \frac{s_0}{\Delta} \qquad (715)$$

in which s_0 is the mean semidiameter and Δ the actual distance of the planet from the earth, the semi-major axis of the earth's orbit being unity. To find the value of s_0 from the values of s observed at different times, we have then only to take the mean of all its values found by the formula

$$s_0 = s\Delta \qquad (716)$$

taking Δ from the tables of the planet for each observation.

But here it is to be remarked that, in micrometric measures of the apparent diameter of a planet, different values will be obtained by different observers or with different instruments. The spurious enlargement of the apparent disc arising from imperfect definition of the limb, or from the *irradiation* resulting from the vivid impression of light upon the eye, will vary with the telescope, and may also vary for the same telescope when eye pieces of different powers are employed. The irradiation may be assumed to consist of two parts, one of which is constant and the other proportional to the semidiameter. Those errors of the observer which are not *accidental* may also be supposed to consist of two parts, one constant and the other proportional to the semidiameter; the first arising from a faulty judgment of a contact of a micrometer thread with the limb of the planet, the second, from the variations in this judgment depending on the magnitude of the disc observed, and possibly also upon any peculiarity of his eye by which the irradiation is *for him* not the same quantity as for other observers. With the errors proportional to the semidiameter will be combined also any error in the supposed value of a revolution of the micrometer. The errors of the two kinds will, however, be all represented in the formula

$$s_0 = (s + x + sy)\Delta \qquad (717)$$

where x is the sum of all the constant corrections which the

observed value s requires, and sy is the sum of all those which are proportional to s. Now, let

$$s_1 = \text{an assumed value of } s_0,$$
$$ds_1 = \text{the unknown correction of this value}$$
$$s_0 = s_1 + ds_1;$$

then the above equation may be written

$$0 = s\mathit{\Delta} - s_1 + x\mathit{\Delta} + sy\mathit{\Delta} - ds_1$$

But $sy\mathit{\Delta}$ will be sensibly the same as $s_0 y$. It will, therefore, be constant, and will combine with ds_1. We shall, therefore, put z for $sy\mathit{\Delta} - ds_1$, and then, putting

$$n = s\mathit{\Delta} - s_1$$

our equations of condition will be of the form

$$x\mathit{\Delta} + z + n = 0 \qquad\qquad (718)$$

from all of which x and z may be found by the method of least squares. But it will be impossible to separate the quantity ds_1 from z; we can only put

$$(s_0) = s_1 - z$$

whereas we have, for the true value,

$$s_0 = s_1 + ds_1 = s_1 - z + s_0 y$$

or

$$s_0 = (s_0) (1 + y) \qquad\qquad (719)$$

and then, if any independent means of finding y are discovered, the true value of s_0 can be computed.

THE ABERRATION CONSTANT AND THE ANNUAL PARALLAX OF FIXED STARS.

436. The constant of aberration is found by (669) when we know the velocity of light and the mean velocity of the earth in its orbit. The progressive motion of light was discovered by ROEMER, in the year 1675, from the discrepancies between the predicted and observed times of the eclipses of Jupiter's satellites. He found that when the planet was nearest to the earth the eclipses occurred about 8^m earlier than the predicted times, and when farthest from the earth about 8^m later than the predicted

times. The planet was nearer the earth in the first position than in the second by the diameter of the earth's orbit; and hence ROEMER was led to the true explanation of the discrepancy,— namely, that light was progressive and traversed a distance equal to the diameter of the earth's orbit in about 16^m. More recently, DELAMBRE, from a discussion of several thousand of the observed eclipses, found $8^m 13^s.2$ for the time in which light describes the mean distance of the earth from the sun. From this quantity, which is denoted by $\dfrac{a}{V}$, Art. 395, we obtain the aberration constant by the formula

$$k = \frac{a}{V} \cdot \frac{2\pi}{n\, T \sin 1'' \sqrt{1 - e^2}} \qquad (720)$$

Hence, with the values $\dfrac{a}{V} = 493^s.2$, $T = 366.256$, $n = 86164$, $e = 0.01677$, we find $k = 20''.260$. DELAMBRE gives $20''.255$, which would result from the above formula if we omitted the factor $\sqrt{1 - e^2}$, as was done by DELAMBRE.

On account of the uncertainty of the observations of these eclipses (resulting from the gradual instead of the instantaneous extinction of the light reflected by the satellite), more confidence is placed in the value derived from direct observation of the apparent places of the fixed stars.

437. *To find the aberration constant by observations of fixed stars.*— Observations of the right ascension of a star near the pole are especially suitable for this purpose, because the effect of the aberration upon the right ascension is rendered the more evident by the large factor $\sec \delta$ with which in (678) the constant is multiplied. The apparent right ascension should be directly observed at different times during at least one year, in which time the aberration obtains all its values, from its greatest positive to its greatest negative value. If we suppose but two observations made at the two instants when the aberration reaches its maximum and its minimum, the earth at these times being in opposite points of its orbit, and if α' and α'' are the apparent right ascensions at these times (freed from the effects of the nutation and the precession in the interval between the observations), we shall have

$$k = \tfrac{1}{2} (\alpha' - \alpha'') \cos \delta$$

observations of each year are separately discussed. The period of the leading terms of the nutation being only nineteen years, if we extend the observations for aberration over a considerable portion of this period, it will be proper to introduce into our equations of condition a term involving the correction of the nutation constant, as will be seen hereafter.

438. The declinations may also be employed for determining the aberration. If we put

$\delta =$ the assumed mean declination $+$ the nutation,
$\Delta\delta =$ the correction of this value,
$\delta' =$ the observed value,

we have, by (678),

$$\delta' = \delta + \Delta\delta - (k + \Delta k)\,[(\sin \epsilon \cos \delta - \cos \epsilon \sin \delta \sin a) \cos \odot$$
$$+ \sin \delta \cos a \sin \odot]$$

or, putting

$$m' \sin M' = \sin \delta \cos a$$
$$m' \cos M' = \cos \delta \sin \epsilon + \sin \delta \cos \epsilon \sin a$$

and then

$$a' = - m' \cos (\odot - M')$$
$$n' = \delta + a'k - \delta'$$

the equation of condition is

$$a' \Delta k + \Delta\delta + n' = 0 \qquad (723)$$

439. If the pole star is employed, which has a sensible annual parallax, or any star whose parallax is even suspected, it will be proper to introduce into the equations of condition a term which represents its effect. We have, by (691), introducing the above auxiliaries,

$$\text{par. in R. A.} = + pr\,m \sin (\odot - M) \sec \delta$$
$$\text{par. in dec.} = + pr\,m' \sin (\odot - M')$$

and hence the equation of condition from the right ascension will be

$$a \Delta k + bp + \Delta a + n = 0 \qquad (724)$$

and, from the declination,

$$a' \Delta k + b'p + \Delta\delta + n = 0 \qquad (725)$$

from the right ascensions of the pole star $k = 20''.4486$, and the annual parallax of the star $= 0''.1444$; PETERS, from six hundred and three equations of condition, formed upon the right ascensions of the pole star, observed at Dorpat in the years 1822 to 1838, found $k = 20''.4255$, with the annual parallax $= 0''.1724$; LUNDAHL, from one hundred and two observed declinations of this star, found $k = 20''.5508$, and the parallax $= 0''.1473$; and PETERS, from two hundred and seventy-nine declinations of the same star, observed with the Repsold vertical circle of the Pulkova Observatory, found $k = 20''.503$, and the parallax $= 0''.067*$.

The parallax is so small a quantity that the discrepancies between these several values appear to be relatively great: nevertheless, we must consider them as surprisingly small when we remember that all these determinations rest upon observations of the absolute place of the star. *Differential* measures of the changes of a star's place with the micrometer are susceptible of greater refinement. Such a method I proceed to give in the next article.

441. *To find the relative parallax of two stars by micrometric measures of their apparent angular distance.*—It was first suggested by the elder HERSCHEL that if the absolute linear distances of two neighboring stars from our solar system were very unequal, their apparent angular distance from each other as seen from the earth would necessarily vary as the earth changed its position in its orbit. If one of the stars were so remote as to have no sensible parallax, changes in this apparent distance (provided they followed the known law of parallax) might be ascribed solely to the parallax of the nearer star; and in any case such changes might be ascribed to the relative parallax; that is, to the difference of the parallaxes of the two stars.

For the trial of this method BESSEL judiciously selected the star 61 *Cygni*, near which are two much smaller stars (at distances from it of about 8′ and 12′ respectively), and from a series of micrometric measures of its angular distance from each, extending through a period of more than a year, namely, from August 18, 1837, to October 2, 1838, obtained the first clearly demonstrated parallax of a fixed star.† A subsequent

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* *Astron. Nach.*, Vol. XXII. p. 119.   † Ibid. No. 366.

distance AB or $A'B - AB$, which will be denoted by Δs, is given by the differential formula

$$\Delta s = - \Delta a \cos \delta . \sin P - \Delta \delta \cos P$$

where Δa and $\Delta \delta$ are respectively the parallax in right ascension and declination, which are given by (691). Substituting these values, and then assuming the auxiliaries m and M, such that

$$m \cos M = \quad \sin a \sin P + \cos a \sin \delta \cos P$$
$$m \sin M = (- \cos a \sin P + \sin a \sin \delta \cos P) \cos \epsilon$$
$$- \cos \delta \cos P \sin \epsilon$$

we have

$$\Delta s = prm \cos (\odot - M) \qquad (726)$$

The effect of the proper motion of A upon the distance is found as follows. Let

$\chi =$ the angle which the great circle in which the star moves makes with the declination circle,

$\rho =$ the annual proper motion on the great circle,

$\Delta'a, \Delta'\delta =$ the given proper motion in right ascension and declination, reduced to the assumed epoch (Art. 379);

then, as in Art. 380, we find ρ and χ by the formulas

$$\left. \begin{array}{l} \rho \sin \chi = \Delta'a \cos \delta \\ \rho \cos \chi = \Delta'\delta \end{array} \right\} \quad (727)$$

Let τ be the time of any observation reckoned from the assumed epoch and expressed in fractional parts of a year. In the above diagram, if AA' now represents the proper motion on a great circle in the time τ, then $AA' = \tau\rho$; and, if the effect of the proper motion upon the distance is denoted by $\Delta's$, we have also $A'B = s + \Delta's$, $A'AB = P - \chi$, and the triangle $AA'B$ gives

$$\cos (s + \Delta's) = \cos (\tau\rho) \cos s + \sin (\tau\rho) \sin s \cos (P - \chi)$$

Developing this equation, and retaining only second powers of $\tau\rho$, we find

$$\Delta's = - \tau\rho \cos (P - \chi) + \frac{(\tau\rho)^2 \sin^2 (P - \chi)}{2s}$$

in which τ is the only variable.   Taking then for the constants

$$\left.\begin{array}{l} f = - \rho \cos{(P - \chi)} \\ f' = \dfrac{\rho^2 \sin^2(P - \chi)}{2s} \end{array}\right\} \quad (728)$$

the computation of the correction for each observation is readily made by the formula

$$\Delta's = f\tau + f'\tau\tau$$

The assumed proper motion may, however, be in error; and there may also be errors in the observed distances which are proportional to the time (such as any progressive change in the value of the micrometer screw, &c.).   The correction for all such errors may be expressed by a single unknown correction y of the coefficient f, so that we shall take

$$\Delta's = (f + y)\,\tau + f'\tau\tau \qquad (729)$$

The corrections of micrometric measures for the effects of aberration and refraction* are treated of in Vol. II. Chapter X. We shall, therefore, suppose these corrections to have been applied, and shall take

$s' =$ the observed distance at the time τ, corrected for differ-
    ential aberration and refraction,

and then we shall have

$$s' = s + \Delta s + \Delta's \qquad (730)$$

This equation involves three unknown quantities, namely, the distance s, the parallax involved in Δs, and the correction y involved in $\Delta's$.   Let s_0 be an assumed value of s nearly equal to the mean of the values of s', and put

$$s = s_0 + x$$

The substitution of this in our equations of condition will introduce the small unknown quantity x in the place of the larger

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* These effects are only differential, and so small that the errors in the total refraction and aberration may safely be assumed to have no sensible influence.   It is also an advantage of this method of finding the parallax of a star, that it is free from the errors of the nutation and precession, which, being only changes in the position of the circles of reference, have no effect whatever upon the apparent distance of two stars.

one s, and will thus facilitate the computations. When all the substitutions are made in the expression of s', we obtain the following equation:

$$0 = s_0 - s + f\tau + f'\tau\tau + x + \tau y + prm \cos (\odot - M)$$

To put this in the usual form, let us take

$$n = s_0 - s' + f\tau + f'\tau\tau$$
$$c = rm \cos (\odot - M)$$

then each observation gives the equation

$$x + \tau y + cp + n = 0 \qquad (731)$$

and from all these equations we find, by the method of least squares, the most probable values of x, y, and p.

In the determination of so small a quantity as p, it is necessary to give to the micrometric measures the greatest possible precision. It is particularly important to find the effects of temperature upon the micrometer screw; for these effects, depending on the season, have a period of one year, like the parallax itself, and may in some cases so combine with it as completely to defeat the object of the observations. At the time BESSEL published his discussion of his observations on 61 *Cygni*, he had not completed his investigations of the effect of temperature upon the screw, and therefore introduced an indeterminate quantity k into his equations of condition, by which the effect upon the parallax might be subsequently taken into account when the correction for temperature was definitively ascertained. This was done as follows. He had assumed the correction of a measured distance for the temperature of the micrometer screw to be

$$\Delta''s = - 0''.0003912\, s\, (t - 49°.2)$$

in which t is the temperature by Fahrenheit's scale, and s is expressed in revolutions of the screw. If the coefficient $0''.0003912$ should be changed by subsequent investigations to $0''.0003912 \times (1 + k)$, each observed distance would receive the correction $\Delta''s.k$, the quantity n in the equations of condition would become $n - \Delta''s.k$, and the equations would take the form

$$x + \tau y + cp - \Delta''s.k + n = 0 \qquad (732)$$

of an assumed aberration and nutation, and denote these apparent values by α and δ, and put

$\Delta\nu =$ the correction of the nutation constant,
α', $\delta' =$ the observed right ascension and declination;

then

$$\alpha' = \alpha + \Delta\alpha + a\,\Delta k + b\,p + c\,\Delta\nu$$
$$\delta' = \delta + \Delta\delta + a'\Delta k + b'p + c'\Delta\nu \qquad (733)$$

in which, as before, $\Delta\alpha$ and $\Delta\delta$ are the corrections of the star's mean place, Δk the correction of the aberration constant, p the star's annual parallax, a and b, a' and b' are the coefficients found in Arts. 437, 438, and 439. It only remains to express c and c' in terms of known quantities.

In the physical theory, it is shown that the coefficients of those terms of the nutation formulæ (666) which depend upon $2\odot$, $\odot - \Gamma$, and $\odot + \Gamma$ involve not only the nutation constant (the coefficient of $\cos \Omega$), but also the precession constant; while all the other coefficients vary proportionally to the coefficient of $\cos \Omega$. If we put

$\nu =$ the assumed nutation constant,
$\nu' =$ the true   "   " $= \nu + \Delta\nu$

and if we express the relation between ν and ν' by the equation

$$\nu' = \nu\,(1 + i)$$

and, in like manner, suppose the true precession constant to be

$$\psi = 50''.3798\,(1 + \zeta)$$

then, according to PETERS,[*] the formulæ (666), adapted for any value of the constants, are for 1800,

$$\Delta\varepsilon = (1 + i)\,[9''.2231 \cos \Omega - 0''.0897 \cos 2\Omega + 0''.0886 \cos 2\mathbb{C}\,]$$
$$+ (1 - 2.162\,i + 3.162\,\zeta)\,[0''.5510 \cos 2\odot + 0''.0093 \cos\cdot(\odot + \Gamma')]$$
$$\Delta\lambda = (1 + i)\,[-17''.2405 \sin \Omega + 0''.2073 \sin 2\Omega - 0''.2041 \sin 2\mathbb{C}$$
$$- 0''.0677 \sin (\mathbb{C} - \Gamma')]$$
$$+ (1 - 2.162\,i + 3.162\,\zeta)\,[-1''.2694 \sin 2\odot + 0''.1279 \sin (\odot - \Gamma)$$
$$- 0''.0213 \sin (\odot + \Gamma')]$$

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[*] *Numerus Constans Nutationis*, p. 46. We have omitted some terms which are inappreciable or of very short period. This omission will not affect the accuracy of the determination of the quantity ν.

1747, and embracing, therefore, a whole period of the nutation, found $k = 20''.2116$, $\nu = 9''.2320$. In this discussion the parallax of the stars was not taken into account.

Nearly the same value of the nutation constant follows from the more recent observations at the Pulkova Observatory. From the declinations of the pole star observed between 1822 and 1838, LUNDAHL found $\nu = 9''.2164$, and from the right ascensions of the same star PETERS found 9''.2361. The value 9''.2231, which PETERS has adopted in the *Numerus Constans Nutationis*, is the mean of the three values found by BUSCH, LUNDAHL, and himself, having regard to the weights of the several determinations as given by their probable errors.

THE PRECESSION CONSTANT.

443. If α_1, δ_1, and α_2, δ_2 are the mean right ascensions and declinations of the same star, deduced from observation at two distant epochs t_1 and t_2, by deducting from the observed values the aberration and nutation, the *annual variations* of the right ascension and declination for the mean epoch $\frac{1}{2}(t_1 + t_2)$ will be

$$a = \frac{\alpha_2 - \alpha_1}{t_2 - t_1} \qquad b = \frac{\delta_2 - \delta_1}{t_2 - t_1} \qquad (735)$$

These annual variations include both the precession and the proper motion of the star; and, since both are proportional to the time, it will be impossible to distinguish the proper motion until the precession is obtained. If, however, we suppose that the proper motions of the different stars observe no law, or that they take place indiscriminately in all directions, it will follow that the mean value of the precession, deduced from such annual variations of a very large number of stars, will be free from the effect of the proper motions. The latter are, in fact, so various in direction, although, as will hereafter be shown, not entirely without law, that this mode of proceeding must lead at least to an approximation not very far from the truth. Accordingly, from the a and b, found as above for each star, we derive the m and n of Art. 374, by the equations*

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* Both m and n may be found from the right ascensions alone by forming equations of the form

$$m + n \sin \alpha_0 \tan \delta_0 = a$$

from a number of stars and solving them by the method of least squares.

THE MOTION OF THE SUN IN SPACE.

444. With a knowledge of the precession we are enabled to distinguish proper motions in a large number of stars. Upon comparing these proper motions, Sir W. HERSCHEL was the first to observe that they were not without law, that they did not occur indiscriminately in all directions, but that, in general, the stars were apparently moving *towards* the same point of the sphere, or *from* the diametrically opposite point. The latter point he located near the star λ *Herculis*. This common apparent motion he ascribed to a real motion of our solar system, a conclusion which has since been fully confirmed.

Nevertheless, there are many stars whose proper motions are exceptions to this law: these must be regarded as motions compounded of the real motions of the stars themselves and that of our sun. These real motions must, doubtless, also be connected by some law which the future progress of astronomy may develop;* but thus far they present themselves in so many directions that (like the whole proper motion in relation to the precession) they may be provisionally treated as accidental in relation to the common motion. Hence, for the purpose of determining the common point from which the stars appear to be moving, and towards which our sun is really moving, we may employ *all* the observed proper motions, upon the presumption that the real motions of the stars, having the characteristics of accidental errors of observation and combining with them, will be eliminated in the combination. Nevertheless, in order that the errors of observation may not have too great an influence, it will be advisable to employ only those proper motions which are large in comparison with their probable errors.

The direction in which a star appears to move in consequence of the sun's motion lies in the great circle drawn through the star and the point towards which the sun is moving. Let this point be here designated as the point O. If the great circle in which each star is observed to move were drawn upon an artificial globe,

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* The law which we naturally expect to find is that of a revolution of all the stars of our system around their common centre of gravity. MÄDLER, conceiving that our knowledge of the proper motions is already sufficient for the purpose, has attempted to assign the position of this centre. He has fixed upon *Alcyone*, the principal star of the *Pleiades*, as the central sun. *Astron. Nach.*, No. 566. *Die Eigenbewegungen der Fixsterne in ihrer Beziehung zum Gesammtsystem*, von J. H. MÄDLER, Dorpat, 1856.

angle χ produced by the variations of A and D will be found from the triangle POS by the first differential formulæ (47); whence

$$\Delta\chi . \sin\lambda = (\chi'-\chi)\sin\lambda = \left(\frac{\cos(a-A)\cos\delta\sin D - \sin\delta\cos D}{\sin\lambda}\right)\Delta A\cos D$$

$$+ \frac{\sin(a-A)\cos\delta}{\sin\lambda}\Delta D \qquad (740)$$

Hence, we have only to compute for each star the values of χ and $\sin\lambda$ by (738), and of χ' by (739), and then, putting

$$n = (\chi - \chi')\sin\lambda$$

$$a = \frac{\cos(a-A)\cos\delta\sin D - \sin\delta\cos D}{\sin\lambda}$$

$$b = \frac{\sin(a-A)\cos\delta}{\sin\lambda}$$

we form the equation of condition,

$$a . \Delta A\cos D + b . \Delta D + n = 0$$

in which $\Delta A\cos D$ and ΔD are the unknown quantities. From all the equations thus formed the most probable values of ΔA and ΔD will be found by the method of least squares. The quantity $(\chi - \chi')\sin\lambda$ is the distance between the great circle in which the star really moves and that drawn from the star to the point O, measured at this point.

In this manner the position of the point O has been very closely determined. The earlier determinations founded on a comparatively small number of well established proper motions are those of

W. HERSCHEL, $A = 245°\ 53'$ $D = +49°\ 38'$
and   GAUSS, $A = 259\ 10$ $D = +30\ 50$

Of the more recent determinations, the first in the order of time is that of ARGELANDER.[*] He employed 390 stars, the proper motions of which he found by comparing their positions as determined by himself for 1830[†] with those determined by BESSEL from BRADLEY's observations for 1755.[‡] He divided these stars into

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* *Astron. Nach.*, No. 863.   † DLX *Stell. Fix. Positiones Mediæ ineunte anno* 1830.
‡ *Fundamenta Astronomiæ.*

than in the other; and the aggregate of all the proper motions, having regard to their signs, would not be zero.

Since it is probable that the average distance of stars of the *same magnitude* is the same on all sides of us (although there are not a few individual exceptions of small stars with large proper motions and large stars with small ones), a more satisfactory determination of the precession constant may result from future investigations in which not only all the stars employed shall be uniformly distributed, but those of each order of apparent magnitude shall be so distributed. It will be impossible to secure this condition if the larger stars are retained; for their distribution is too unequal. By confining the investigation to the small stars, there will also be obtained the additional advantage that the amount of the proper motions themselves will probably be very small, and thus have very little influence upon the precession constant, even if they are not wholly eliminated. The formation of accurate catalogues of the small stars is therefore essential to the future progress of astronomy in this direction.

END OF VOL. I.